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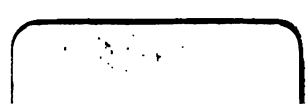
SOLID GEOMETRY

W. TIMBRELL PIERCE



PRACTICAL SOLID

OR

DESCRIPTIVE GEOMETRY.

TREATISE

ON

PRACTICAL SOLID

OR

DESCRIPTIVE GEOMETRY:

EMBRACING

ORTHOGRAPHIC PROJECTION

AND

PERSPECTIVE OR RADIAL PROJECTION.

BY

W. TIMBRELL PIERCE, ARCH$^{T.}$

LATE LECTURER ON GEOMETRICAL DRAWING AT KING'S COLLEGE, LONDON (VICE PROFESSOR BRADLEY), AND HARROW SCHOOL.

WITH 85 PLATES OF ORIGINAL DRAWINGS.

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183. h. 26.

TO HIS BROTHER,

JAMES PIERCE, M.A.

OF CORPUS CHRISTI COLLEGE, CAMBRIDGE;
MATHEMATICAL MASTER OF BEDFORD GRAMMAR SCHOOL;

This Work is Affectionately Dedicated

BY

THE AUTHOR.

INTRODUCTION.

I HAVE BEEN INDUCED to write this work on Practical Solid Geometry from the want felt by all English Students of a good text-book on the subject. Although many excellent works exist in foreign languages, I am not acquainted with any thoroughly good English book.

Probably this may be accounted for from the fact that on the Continent the subject is more generally and systematically studied; but, as technical education is becoming more and more appreciated in this country, a good text-book on Geometrical Drawing will be in great request. I have made ample use of existing works: among others of recent date I would mention that by the late Professor Bradley, which is the most able, but the text is not sufficiently full or explicit to enable a learner to study it by himself without other help. I am also largely indebted to Leroy's 'Traité de Géométrie descriptive' and Hamilton's 'Stereography.'

In a future work I propose to show the application of the subject to the several arts of Construction.

ROEHAMPTON: 1873.

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DESCRIPTIVE GEOMETRY.

PART I.

ORTHOGRAPHIC PROJECTION.

CHAPTER I.

DEFINITIONS, ETC.—PROJECTIONS OF POINTS AND LINES.

1. *Descriptive Geometry* is the science which enables us to represent on a plane surface solid objects.

2. Hence its application to all the arts of construction, such as architecture, engineering, fortification, &c.; to artists, also, a knowledge of this subject is most important (though not so generally recognised), as enabling them to draw in true perspective, cast the shadows of objects truly, and obtain the true position of the locus of greatest light on an object; problems, some of them of great nicety, the solutions of which are for the most part due to the geometrical genius of the great French mathematician Monge, who indeed may be regarded as the first to have developed the general principles of the subject, and presented it in a systematic form.

3. Those who are acquainted with co-ordinate Geometry will have no difficulty in understanding that the position of a solid is fixed in space, when we know its projections on certain fixed planes, called co-ordinate planes, or more commonly in this subject, *planes of projection*.

4. These *planes of projection*, for the sake of simplicity, are supposed to be at right angles to each other, and one is assumed to be horizontal and the other vertical.

5. The line of intersection between the planes of projection is termed the *ground line*. It is the line from which the distances of points in space from either plane are to be measured.

6. Thus let L O M, N O P represent the two planes of projection taken at right angles to the plane of the paper. Next suppose the plane L O M to revolve about the line of intersection or ground line (here projected in the point O, being perpendicular to the paper), until it coincide with N O P. The two planes will then be confused together, and present a plane surface of indefinite extent upon which to represent the projections of the solid required.

7. The plane of the paper, therefore, on which the projections of any solid are drawn, represents the two planes of projection—the line of intersection or ground line being always drawn, from which to measure the distances of the points in space from the two planes of projection.

8. *Note.*—Throughout the rest of this work the following notation will be adopted as far as is practicable:—When reference is made to the actual points in space, we shall distinguish them by capital letters, as A, B, C, &c. When we refer to their horizontal projections, or their projections on the horizontal plane of projection, we shall use the small letters a, b, c, &c.; and when to their vertical projections, the small letters accented, as a', b', c', &c. Of course, the same method will apply to all lines and solids, so that when we refer to a line A B, we mean the line in space which has $a\ b$ for its horizontal projection, and $a'\ b'$ for its vertical projection.

9. We will now show how the projections of a point in space are determined in all positions.

Firstly, let us suppose the point A to be in the quadrant L O P (Fig. 1) formed by the two planes of projection. From A drop the perpendicular A a on the horizontal plane N O P, then the foot of this perpendicular (a) is the horizontal projection of the point. In like manner drop the perpendicular A a' on to the vertical plane L O M—the foot of this perpendicular (a') is the

vertical projection of the point. Now, when the plane L O M revolves about the line of intersection O, and coincides with N P O, a' will describe a quarter of a circle; and the straight line $a\ a'$, joining the two points of projection, will be at right angles to the line of intersection O, because it will be the intersection of the vertical projecting plane a' A a with the horizontal plane N O P; and this vertical projecting plane being at right angles to both the planes of projection (since it contains the two perpendiculars A a', A a), will intersect their line of intersection O at right angles. Hence (in Fig. 2), representing the ground line O by X Y, if we make the distance O a below the ground line and at right angles to it, equal to O a or A a' in Fig. 1, (a) will be the horizontal projection of the point, and O a will represent the distance of the point A from the vertical plane of projection. Again, by producing a O above X Y to a' until O a' is equal to O a' or a A in Fig. 1, a' will be the vertical projection of the point, and O a' will represent the height or distance of the point A above the horizontal plane. Hence, when a point is in the first quadrant, or in front of the vertical and above the horizontal plane of projection, its horizontal projection will always be below X Y, and its vertical above. And this must always be remembered—namely, wherever the point may be, the line joining its two projections will be at right angles to the ground line.

Secondly, let A be in the quadrant L O N, and construct the two projections $a\ a'$, as in the first case. It now appears that a and a' are both of them above X Y. Therefore, when a point has its two projections above the ground line, it will be above the horizontal and behind the vertical plane of projection.

Thirdly, when A is in the quadrant N O M, or below the horizontal and behind the vertical, the horizontal projection a will be above the ground line, and its vertical below, as will appear manifest on referring to the figure.

Lastly, let A be in the quadrant M O P. It will then appear that a and a' both lie below the ground line.

There are besides three other positions of the point A which deserve notice.

Firstly, it may lie in the horizontal plane either in front of or behind the vertical, in which case a', the vertical projection, will be in the ground line, and a either below or above the ground line accordingly; or secondly, it may lie in the vertical plane either above or below the horizontal, in which case a will be in the ground line, and a' either above or below the ground line; or, lastly, the point A may lie in both planes at once, in which case, of course, a and a' will coincide with the point itself, and lie on the ground line.

10. The above are all the possible positions a point can assume, and it will be well for the student, before proceeding farther, to render himself perfectly familiar with them, so that when given the projections of a point, he may be able to tell at a glance its exact position in space with respect to the two planes of projection.

11. We will now proceed to the consideration of the projections of lines.

The projections of curved lines in space are determined by finding the projections of separate points lying near together in the curve, and the lines joining them will be the projections required.

12. The projections of straight lines are determined by finding the projections of any two points in the line, and the straight lines joining them will be the projections required.

13. In the case of a point it has been seen that when one projection is given, the other must lie on the perpendicular drawn through the given projection to the ground line; but in the case of straight lines, any two straight lines drawn on the paper (provided only that neither is perpendicular to the ground line) will represent the projections of a straight line in space, which will be the line of intersection of the two projecting planes, one being perpendicular to the horizontal and the other to the vertical plane of projection.

14. When a straight line is perpendicular to either of the planes of projection, its projection on the plane to which it is perpendicular will be a point, being the point where the line produced meets the plane; and its projection on the other plane will be a straight line at right angles to the ground line; both these cases are shown in Fig. 3. When a straight line is parallel to both planes at once, its two projections will be parallel to the ground line, as shown in Fig. 3. When both projections lie in a straight line perpendicular to the ground line, neither projection being a point, the straight line in space will be indeterminate, as we have no means of ascertaining the distance of any point from either plane; and recourse must in this case be had to another projection on an auxiliary plane, perpendicular to one plane of projection but inclined at any angle to the other.

15. It is often very useful to know the position of the points where a straight line in space produced meets the two planes of projection. These points are

termed *traces*, and are distinguished by being called the *horizontal trace*, or the *vertical trace*, according as the line cuts the horizontal or vertical plane of projection.

The problem, therefore, is as follows:

(Plate I., Fig. 4).—Having given the straight line A B, to find its traces.

First, to find its horizontal trace. Since this is the point where A B meets the horizontal plane, it can be no distance from this plane and therefore, its vertical projection must lie in the ground line X Y (9); but it must also lie in the vertical projection of the line a' b'; it must therefore lie in the point t', where these two lines intersect, and the horizontal trace is now determined by finding the horizontal projection of t', by drawing the perpendicular t' t to cut a b, produced in t. This will be the horizontal trace required. By similar reasoning, to find the vertical trace, we produce the horizontal projection to meet the ground line in v, and then erect the perpendicular v v' to cut a' b' in v', the vertical trace required.

16. The converse of this, having given the two traces to determine the line, may now be solved. Let t and v' be the traces of the line required. From t draw t t, perpendicular to the ground line, and meeting it in t,; join t, v', it will be the vertical projection of the line required. Again, draw v' v perpendicular to the ground line and meeting it in v; join t v, this will be the horizontal projection of the line.

17. Having given the projections of a line, to determine its real length.

If a line is parallel to either plane of projection, its projection on that plane to which it is parallel will be equal to the length of the line itself; but if it be inclined at any angle to a plane its projection on that plane will be equal to the length of the line in space multiplied by the cosine of the angle of inclination. Now, to find the true length of the line A B in Fig. 4; suppose A B to revolve about the vertical line B b, always retaining the same inclination to it, until it becomes parallel to the vertical plane of projection. The extremity of the line A will describe a horizontal circle, whose horizontal projection is the circle a a described with centre b and radius b a; and when the line is parallel to the vertical plane of projection its horizontal projection will be the line b a, drawn through b parallel to the ground line to meet the circle a a in a. Now, since the circle a a is horizontal, its vertical projection will be the straight line a' a' drawn through a' parallel to the ground line; and the point a' is now found

as the vertical projection of a; join b' a', it will be the vertical projection of A B now parallel to the vertical plane of projection, and hence will represent its true length.

This same length might also be found by considering the vertical plane projecting A B on to the horizontal plane to be turned down on the line $a\,b$ as a hinge line, until it becomes horizontal. The points A and B will, in this case, describe vertical circles, their horizontal projections being the straight lines b B, a A drawn at right angles to $a\,b$; now the radii of these circles will be the distances of the points A, B from the horizontal which are determined from the vertical projection; hence make b B equal to O b, and a A equal to O a', and join A B, it will be the true length required.

Note.—If B A be produced to meet $b\,a$ produced in t, t will be the horizontal trace of the line.

18. Through a given point to draw a straight line parallel to a given line.

The projections of parallel lines are themselves parallel, as they are the intersections of the parallel projecting planes with the same co-ordinate planes of projection; hence, all that is necessary is to draw through the vertical projection of the given point a line parallel to the vertical projection of the given line, and through the horizontal projection of the given point a line parallel to the horizontal projection of the given line; these will be the vertical and horizontal projections respectively of the line required.

Cor.—The projections of equal and parallel lines are equal and parallel.

19. The student will now be able to draw the projections of simple solids, when given in certain elementary positions. The following examples are given as studies which the student is recommended to work out for himself on as large a scale as he can.

Example 1.—A pyramid has its axis perpendicular to the vertical plane of projection, which plane contains the base of the pyramid having twelve edges, one of them being inclined at a given angle to the horizontal plane. Draw a plan of the solid and an elevation or vertical projection on a plane inclined to the plane of the base at a given angle.

(Plate II.).—Draw the ground line $x\,y$, and make one of the edges of the base inclined to it at the given angle θ. On this line describe a regular polygon having twelve sides, and join its centre with all the angular points. This will

Fig. 1.

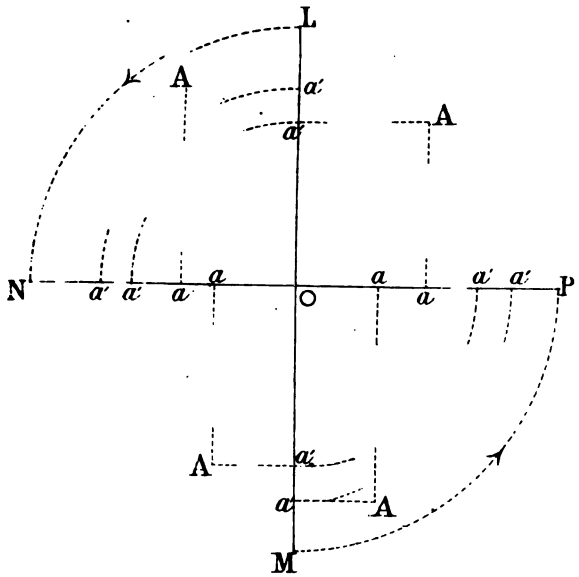


Fig. 2.

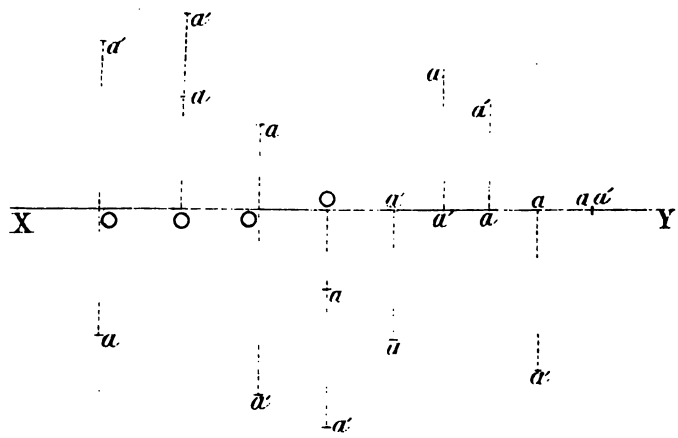


Fig. 3.

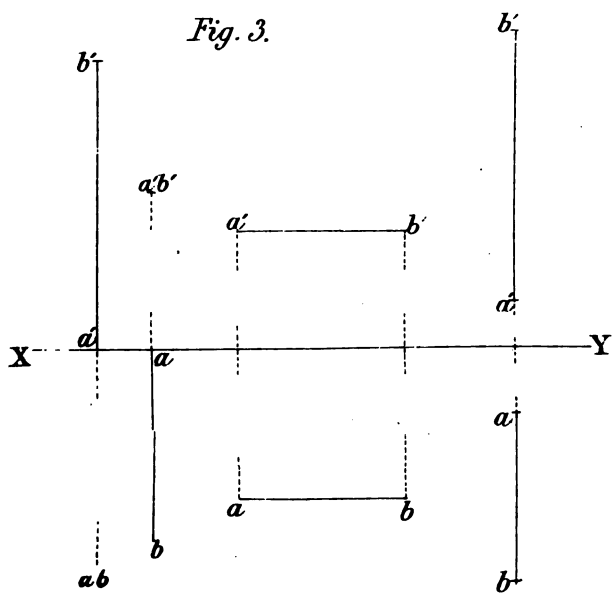
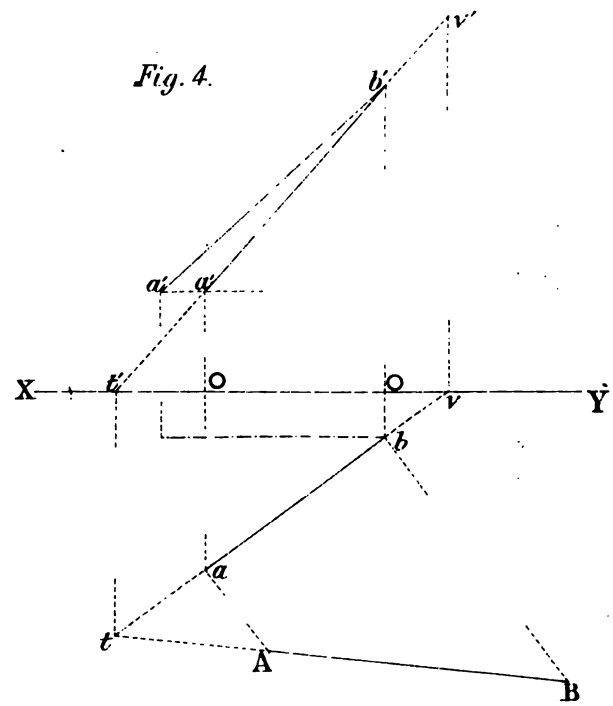
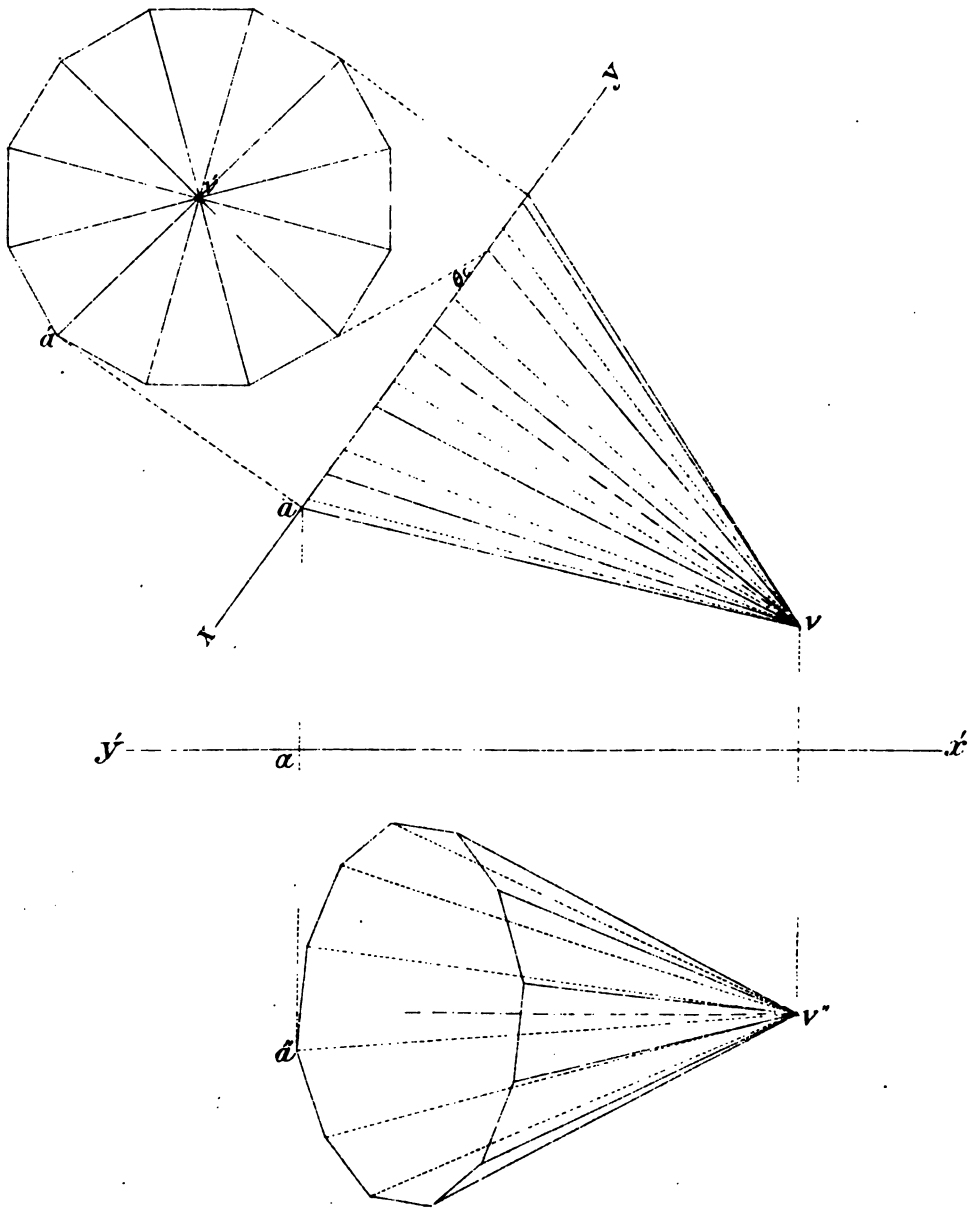


Fig. 4.

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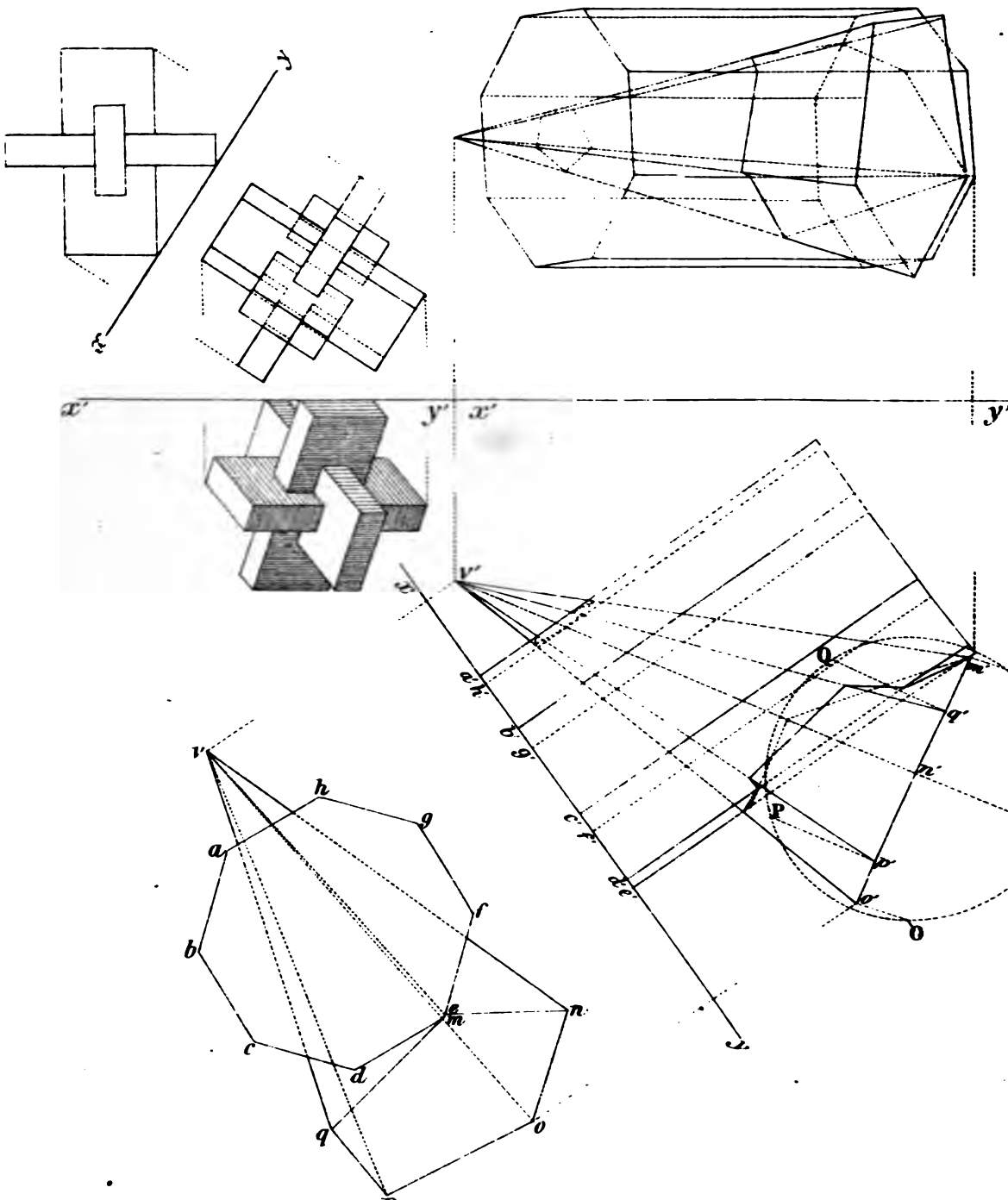
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represent the vertical projection of the solid, having its axis projected in the point v' (14) because it is perpendicular to the plane of projection. Now, in order to determine its plan or horizontal projection, since the base of the pyramid lies in the vertical plane, its horizontal projection will be in the ground line $x\,y$. Therefore, through all the points of the base draw lines to meet the ground line $x\,y$, and perpendicular to it, as $a'\,a$. These will be the horizontal projections of the points of the base; also through v' draw $v'\,v$ perpendicular to the ground line, and produce it beyond to v, equal to the real length of the axis of the solid, which is in this case parallel to the horizontal plane (14), and join v with all the horizontal projections of the points of the base, taking care to dot in all those lines drawn, from the points of the base which are included in the vertical projection between the two extreme dotted lines at right angles to the ground line and the ground line, because these lines, being the projection of the lower half of the solid, will not be visible.

Another vertical projection may now easily be drawn on any plane inclined at a given angle to the plane of the base. Thus, draw $x'\,y'$ inclined to $x\,y$, the plane of the base at the given angle, for a new ground line. Through all the points in the plan draw lines perpendicular to this ground line, and produce them, making the parts produced equal to the distances of the several points from the horizontal plane, which will remain the same as before. Thus, make $a\,a''$ equal to $a\,a'$, and similarly for all the other points, and join the points as in the figure.

The second example in Plate III. represents the projections of three given equal parallelopipeds, having the same centre and cutting each other perpendicularly.

The vertical projection is taken on a plane parallel to one of the solids, which will therefore show their real length, breadth, and thickness. The solids are supposed to rest on the edges of two of the parallelopipeds, and therefore the ground line is drawn through the projections of these lines, in this case points, as they are perpendicular to the plane of projection (14). The plan may now be easily deduced by drawing the projection through all the points of the vertical projection at right angles to the ground line, remembering that the lengths of all lines parallel to the plane of projection are projected in their real lengths.

Another vertical projection is then deduced on the line $x\ y$, exactly in the same way as in the case of the pyramid in the last Example.

Example III. is that of a given prism and pyramid, having their axes horizontal and intersecting, cutting each other. We will leave the student to work

out this Exercise for himself; the projections being determined and the points of intersection found in precisely the same manner as in the first two Examples.

We will now proceed to show how solids may be projected in any given position, but first it will be necessary to show how planes inclined in any direction are represented, and to solve some problems relating to straight lines and planes, first treated abstractedly, and then illustrated by the projection of some solid under the proposed conditions. This will form the subject of the next chapter.

CHAPTER II.

ON STRAIGHT LINES AND PLANES.

20. Planes are represented by means of their *traces*, or the lines in which they cut the co-ordinate planes of projection. These traces are distinguished by the terms horizontal and vertical, according as the plane cuts the horizontal or vertical co-ordinate plane.

21. As a general rule, the two traces of a plane will intersect the ground line in the same point, because three planes in general have one point in common, which will be the point where the three lines of intersection all meet.

22. Planes may be in the following several positions:—

(1.) *Vertical.* (Fig. 1, Plate IV.), as L O M.—In this case, the vertical trace L O is always perpendicular to the ground line, because the horizontal co-ordinate plane X Y is perpendicular to the vertical co-ordinate plane, and also to the given vertical plane L O M, and therefore will cut their line of intersection L O at right angles. The angle the horizontal trace O M makes with the ground line will express the inclination of the given plane with the vertical plane of projection.

(2.) *Horizontal.*—These planes evidently have no horizontal traces, since they never cut the horizontal co-ordinate plane. Their vertical traces are lines parallel to the ground line, their distance from it representing the height or depth of these planes above or below the horizontal plane of projection.

(3.) *Planes perpendicular to the vertical plane of projection.*—These planes have their horizontal traces, as O M (Fig. 2, Plate IV.), always perpendicular to the ground line, because the vertical co-ordinate plane X Y is perpendicular to the horizontal plane of projection and also to the given plane L O M, and is therefore perpendicular to their line of intersection O M. The angle the vertical trace L O makes with the ground line X Y expresses the inclination of the given plane with the horizontal plane.

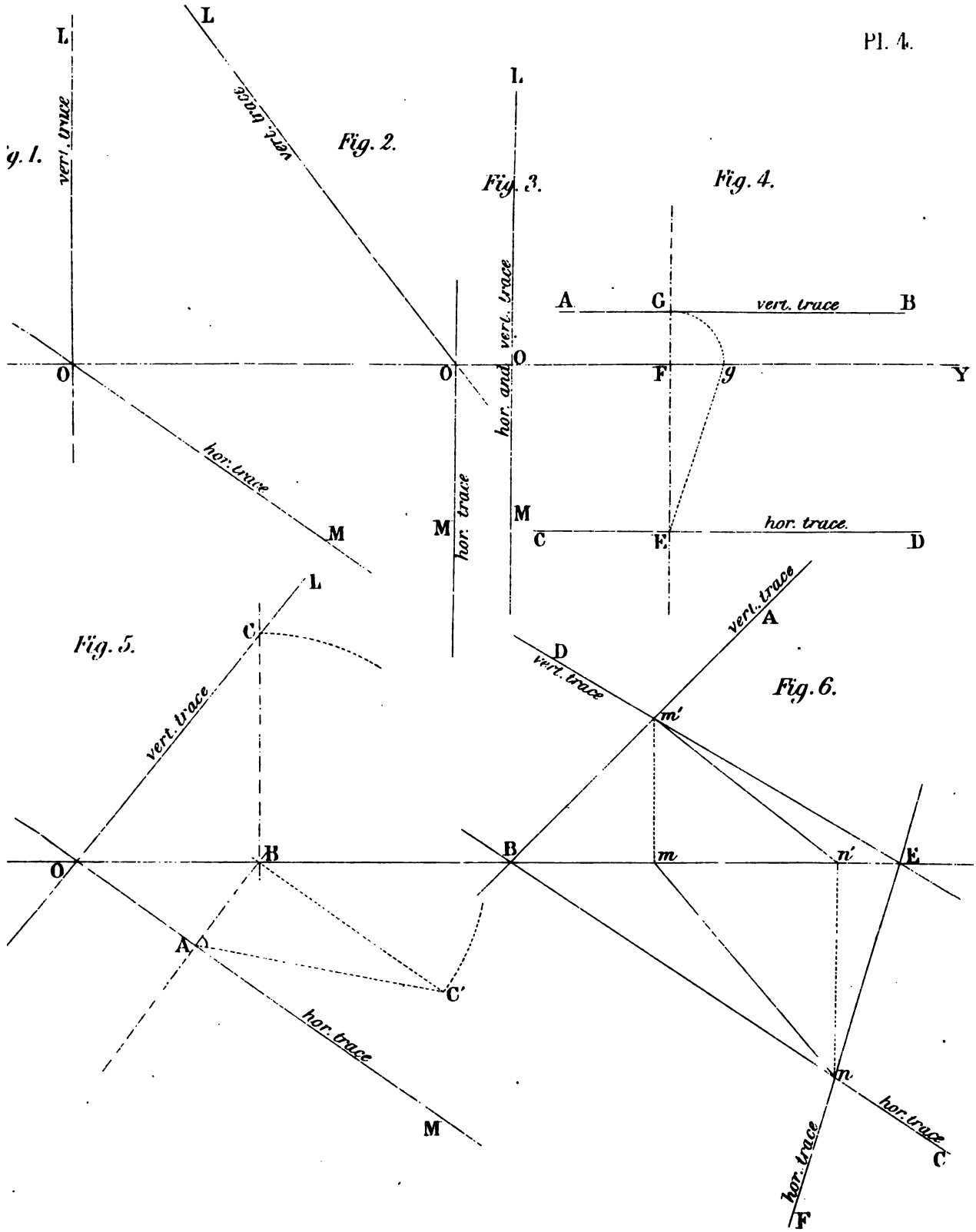
(4.) *Planes perpendicular to both planes of projection.* (Fig. 3, Plate IV.).—The traces of these planes coincide, being both perpendicular to the ground line and meeting it in the same point as L O M in the figure.

(5.) *Vertical planes and parallel to the vertical plane of projection.*—These planes evidently have no vertical traces, as they never cut the vertical co-ordinate plane. Their horizontal traces are lines parallel to the ground line, their distances from it measuring the distance these planes are in front of or behind the vertical plane of projection.

(6.) *Planes inclined to both planes of projection but having their horizontal and vertical traces parallel to the ground line.*—This class of planes forms the principal exception to the general rule laid down in Art. 21. It follows that if a plane is inclined to *both* co-ordinate planes, and has one of its traces parallel to the ground line, the other trace must also be parallel.

Having given the traces of such a plane A B, C D (Fig. 4, Plate IV.), to find its inclinations to the horizontal and vertical planes of projection. Now the inclination of two planes to one another is measured by the angle between the two lines of intersection made by a plane at right angles to both of them and therefore cutting their line of intersection at right angles. In the above case, therefore, let the auxiliary vertical plane E F G be perpendicular to the two traces C D and A B. Since it is perpendicular to both the planes of projection, the two traces will coincide and be perpendicular to the ground line, as in Class IV. The points E and G are evidently points common to this auxiliary plane and the given plane, and therefore the line joining them in space will be the line of intersection required. The height of G is F G, so that by making F g = F G and joining g E, this line will represent the line of intersection turned down into the plane of the paper, and therefore the angle F E g is the angle at which the given plane is inclined to the horizontal. The angle E g F expresses the inclination of the given plane with the vertical plane of projection, because the two angles are complementary. This angle might be found independently by making F e = F E and joining G e; the angle F G e is the angle required.

(7.) *Planes inclined to both co-ordinate planes.*—This class is the most general, and includes all the foregoing. To find the inclination of such a plane as L O M (Fig. 5, Plate IV.), with the horizontal, suppose the vertical plane A B C to be perpendicular to the given plane L O M, A B must therefore be perpendicular to O M. The line in space joining A and C, the two points


g. 1.
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vert. trace
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Fig. 2.
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Fig. 3.
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Fig. 4.
A
G
vert. trace
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C
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Fig. 5.
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Fig. 6.
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common to both the planes will be the line of intersection required. Now the height of C is B C. Therefore by making B C′ = B C, and at right angles to A B, the angle B A C′ will be the angle required.

A similar construction with respect to the vertical co-ordinate plane will give the inclination of the given plane with the vertical co-ordinate plane. We leave this for the student to work out for himself as an exercise; also the following problem:—A plane is inclined to the vertical plane at an angle of 70°, and its vertical trace is inclined to the ground line at an angle of 30°. Find its horizontal trace.

23. Having in the previous Section shown how planes are determined in all possible positions, we will now proceed to the investigation of some elementary problems.

24. Having given the traces of two planes cutting one another, to find the line of intersection (Fig. 5, Plate IV.).—Let A B C, D E F be the two planes. The two horizontal traces meet in n and the two vertical traces in m', hence the line in space joining these two points, which are common to each of the planes, will be the line of intersection required. Since m' is in the vertical co-ordinate plane, its horizontal projection will be (m) in the ground line (9), and therefore $m\,n$ is the horizontal projection of the line of intersection. Again, since n is in the horizontal plane, its vertical projection will be in the ground line (9), and therefore $n'\,m'$ is the vertical projection of the line of intersection.

When either the horizontal or vertical traces of two planes are parallel, the line of intersection will be horizontal or parallel to the vertical co-ordinate plane. In Fig. 1, Plate V., let the two given planes A B C, D E F have their horizontal traces B C, E F parallel. Now the vertical traces intersect in i', the horizontal projection of which, i, is in the ground line; and as the horizontal traces are parallel they may be supposed to meet in a point at an infinite distance, and therefore a line, $i\,l$, drawn through i parallel to A B or E F will be the horizontal projection of the line of intersection. As this line must be horizontal, its vertical projection will be the line $i'\,l'$ drawn through i' parallel to the ground line. The construction is precisely similar in the case where the horizontal traces intersect and the vertical traces are parallel.

When both the horizontal and vertical traces are parallel the planes will generally be themselves parallel, and will therefore have no line of intersection. The reverse of this is always true, viz.:—When planes are parallel their traces

are always parallel. The exception in the first case is where the traces are parallel to the ground line. In this case, the planes *may* not be parallel. Thus in Fig. 2, Plate V., let A B, A′ B′, C D, C′ D′, be the traces of two such planes. Now, in order to find their line of intersection, we must have recourse to the auxiliary vertical plane $m\,o\,g$, and find the two lines of intersection of this plane with the two given planes in the same manner as described in Section 22, Class (6). Let M p, N q represent these two lines turned down about the horizontal trace $s\,q$ into the plane of the paper; the point O, where they intersect, will be a point in the line of intersection of both planes, and when turned back again into the plane $m\,s\,q$ will be projected in r, and the line $i\,l$ drawn through r, parallel to the ground line, will be the horizontal projection of the line of intersection required; its vertical projection is found by making $s\,o$ equal to r O, the height of O, above the horizontal plane, and drawing the line $i'\,l'$ through o, parallel to the ground line. If the lines N q, M p are parallel, the given planes are parallel, and there is no line of intersection.

25. To find the traces of the plane containing three given points not lying in the same straight line.

Let A B C be the three given points (Fig. 3, Plate V.). Since the line joining two of the points, as A B, lies in the plane required, its horizontal trace will be common to the horizontal co-ordinate plane and the required plane, and will therefore be a point in the horizontal trace of the plane. Therefore find the horizontal traces $\alpha\,\gamma$ of any two of the lines, as A B, B C, by the method shown in (15), and the straight line joining them will be the horizontal trace of the required plane. If the horizontal trace r of the remaining line A C be found, it must also lie on this line, and this will serve as a test of the accuracy of the work. To find the vertical trace of the plane, it is only necessary to find the vertical trace of any one of the lines joining the given points, and the straight line joining it and the point M, where the horizontal trace cuts the ground line, will be the vertical trace required. Thus the plane L M N is the plane of the triangle A B C.

26. If it were required to find the real form of the triangle A B C, the method would be to suppose the plane to revolve about its vertical or horizontal trace until it coincided with the vertical or horizontal co-ordinate plane. In the figure we have supposed it to revolve about its vertical trace, as there is

Fig. 1.

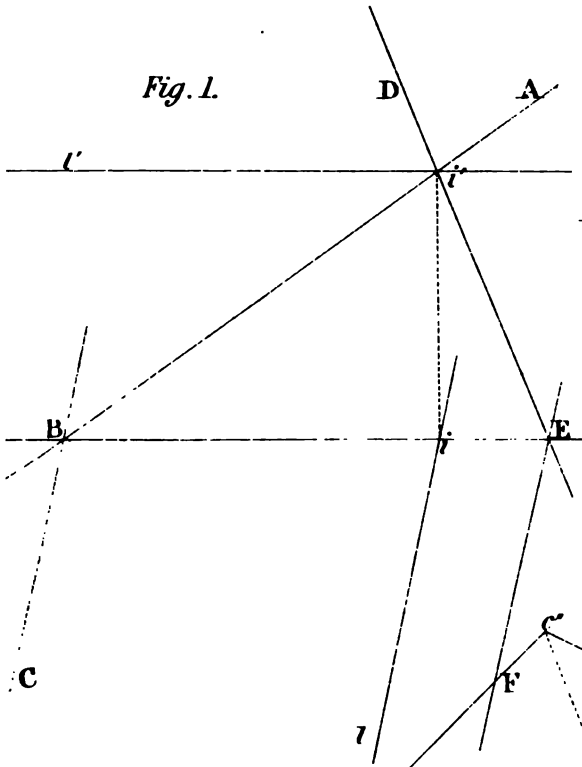


Fig. 2.

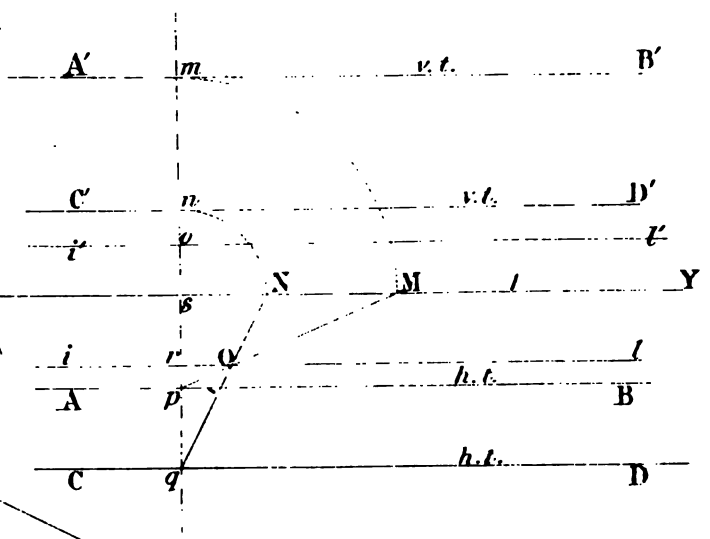
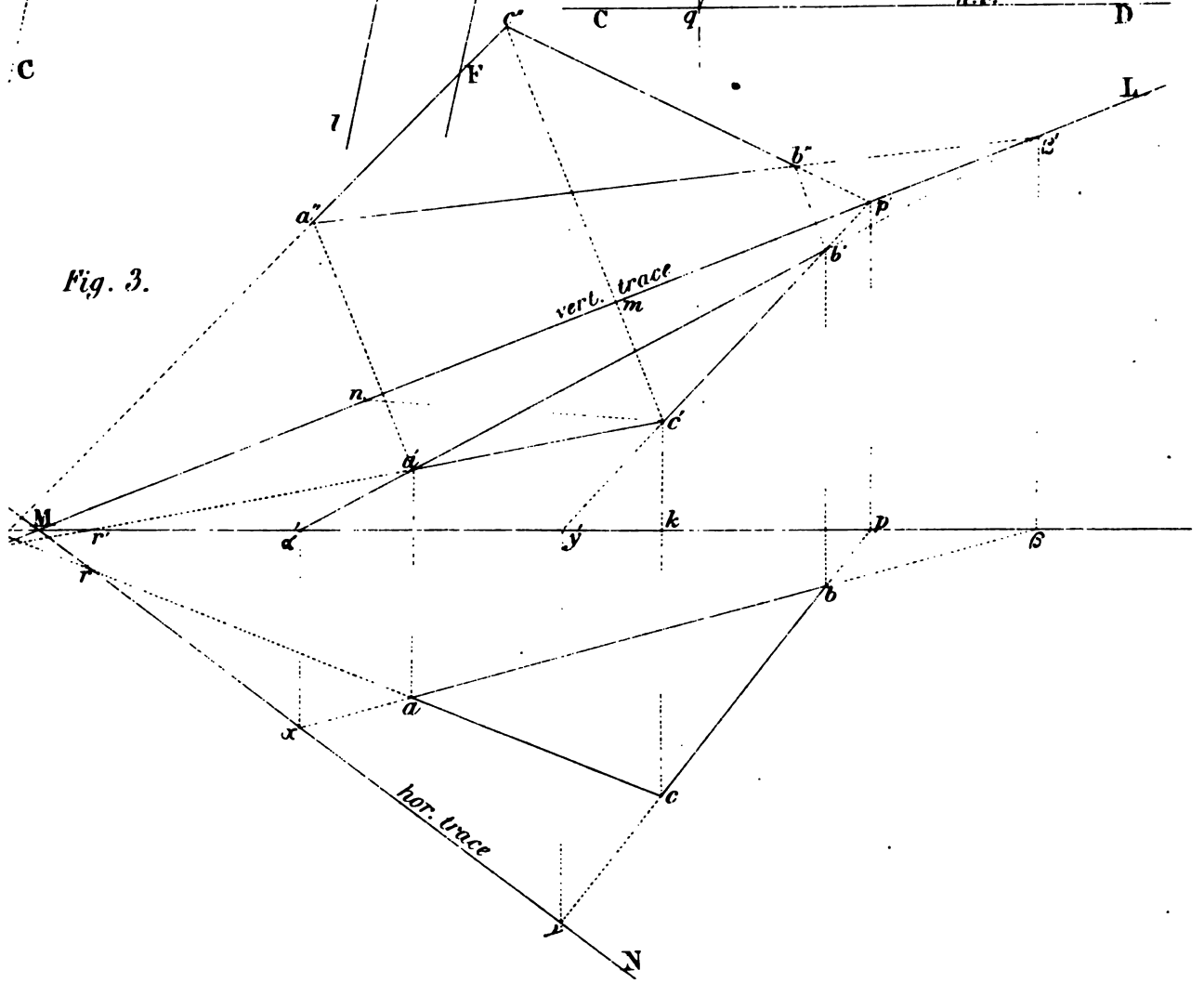


Fig. 3.



more room; but the construction is precisely the same with respect to the horizontal trace.

In this rotation the points A B C will describe circles, having their centres on the vertical trace, and their planes at right angles to it, so that they will be represented by the lines a, $a_{,,}$, c, $c_{,,}$, b, $b_{,,}$ drawn through a, b, and c, at right angles to the vertical trace L M. It only remains to find the distance of the points A B C from the vertical trace, or the radii of the circles described by them. Let us consider the point C. We know that it stands perpendicularly over c' at a distance equal to $k\,c$; and therefore the distance m C is the hypothenuse of a right-angled triangle, of which we know the two sides containing the right angle $m\,c'$ and $k\,c$. Since c, m is at right angles to L M, the shortest way is to set off $m\,n$ on this line equal to $k\,c$, and then the hypothenuse $n\,c'$ will represent the radius of the circle described by the point C, and this distance set off from m to c'' will give the position of the point C when the plane coincides with the vertical plane of projection. The other two points ($a_{,,}$, $b_{,,}$) might be found in a similar manner; but they may be found also more readily as follows:—Since C B meets the vertical trace in p', this point will not move when the plane revolves about L M, as it is a point on the axis of rotation; but we have seen that the point C moves to c'', therefore the line C p, will come into the position $c''\,p'$ and the point b'', where it intersects b, b'', drawn through b' at right angles to L M, will give the position of the point B. Similarly, the position of the point A is found by joining $c''\,a_{,,}$ and finding the point a'' where it intersects the line a, a''.

Thus, $a_{,,}\,b''\,c''$ will represent the true form of the triangle. The above operation is termed *constructing* the figure A B C.

27. To draw a plane through a given point parallel to a given plane.

(Fig. 1, Plate VI.).—Let P be the given point, and L M N the given plane. Since the planes are to be parallel, their horizontal and vertical traces will be respectively parallel; and, since every horizontal line in a plane must be parallel to the horizontal trace of the plane, and that the projections of parallel lines are also parallel, if through p we draw $p\,m$ parallel to the horizontal trace of the given plane N M, it will be the horizontal projection of a horizontal line in the plane required. Its vertical projection will be $p'\,m'$—a line drawn parallel to the ground line. The point m', where this horizontal line meets the vertical co-ordinate plane, will be a point common to that plane and the required plane, and will therefore be a point in the vertical trace of the required line. Through

m', therefore, draw O P parallel to L M, and through P, P Q parallel to M N. O P Q is the plane required.

28. Through a given point to draw a plane at right angles to a given line.

Before solving this problem it will be first necessary to establish the following theorem :—When a straight line is perpendicular to a plane, the projections of the line are respectively perpendicular to the traces of the plane. For if (Fig. 2, Plate VI.), A B is perpendicular to the plane L M N, the projecting vertical plane $a\ b$ is, from its definition, perpendicular to the horizontal plane; it is also perpendicular to the given plane L M N, since it contains the line A B, which is by supposition perpendicular to the given plane. Hence this projecting plane $a\ b$ is perpendicular to both the horizontal plane and the given plane, and is therefore perpendicular to their line of intersection, which is the horizontal trace $m\ n$. Therefore, $a\ b$ is perpendicular to $m\ n$. In like manner, it may be proved that $a'\ b'$ is perpendicular to the vertical trace L M.

The converse of this is also true, viz., if the two projections of a line A B are respectively perpendicular to the traces of a plane L M N, the line A B is itself perpendicular to the plane L M N. For the projecting plane $a\ b$ is evidently perpendicular to the line M N, and therefore to the plane L M N, which contains this line; and the projecting plane $a'\ b'$ is evidently perpendicular to the line L M, and therefore to the plane L M N, which contains this line. Thus the plane L M N is perpendicular to the two projecting planes $a\ b$, $a'\ b'$, and is therefore perpendicular to their line of intersection, which is the line A B in space.

Having established the above important theorem, we shall now find it easy to solve the problem given above. Let P be the given point (Fig. 2, Plate VI.), and A B the given line. It is required to draw a plane through P at right angles to A B.

Since the plane is to be perpendicular to the line, its traces will be perpendicular to the projections of the line. Therefore, if through p we draw $p\ m$ perpendicular to $a\ b$, it will be the horizontal projection of a horizontal line drawn in the required plane; and, since the line is horizontal, its vertical projection will be $p'\ m'$, parallel to the ground line. The point m', where this line meets the vertical co-ordinate plane, will be a point in the vertical trace of the required plane, and it only remains to draw through m' L M at right angles to $a'\ b'$, and through M, where the vertical traces intersect, the ground line M N at right angles to $a\ b$. L M N is the plane required.

29. To find the point of intersection of a given line and a given plane.

(Fig. 3. Plate VI.).—Let A B be the given line, L M N the given plane. Find the line of intersection (24) between the given plane L M N and the vertical plane containing A B, which has for its horizontal trace the line $a\,b$, and, since the plane is vertical, for its vertical trace the line $m\,m'$ at right angles to the ground line. The vertical projection of this line of intersection $n'\,m'$ will contain the vertical projection of the point of intersection of the line A B with the plane L M N; but this point must also lie on the line $a'\,b'$. Therefore, the point b' will be the vertical projection of the point required; and its horizontal projection can now be deduced by projecting b' on to $a\,b$ by the perpendicular to the ground line $b'\,b$.

30. Having given a point P and two straight lines A B and C D, to draw a straight line through P to meet the two given lines (Fig. 4, Plate VI.).

First find the plane which contains P and one of the given lines, as C D. This is done by the method shown in Art. 25. Let $c\,o$ be the horizontal trace, and $o\,d'$ the vertical trace. Next find by the preceding Article where the plane $c\,o\,d'$ meets the line A B. Let I be the point; then the line joining P I will meet C D in K.

Another and neater method of solving this problem is shown in Fig. 1, Plate VII. P is the given point, and A B, C D the given lines. Assume any two points C D on the line C D and join P C, P D; it only remains to find where the plane of the triangle P C D cuts the line A B. To determine this, find the line of intersection of the projecting plane $a'\,b'$ with the plane P C D; the points $e'\,f'$ where $c'\,p'$, $p'\,d'$ intersect $a'\,b'$ will be the vertical projections of two points on this line; the horizontal projections of these points on the line will be found by dropping the perpendicular to the ground line $e'\,e$ and $f'\,f'$ on to $c\,p$ and $d\,p$. Now the line E F will be in the plane P C D, and also in the plane which contains A B, and will therefore meet A B in some point I; the horizontal projection of this point in the point i where $e\,f$ intersects $a\,b$ and its vertical projection i' may now be projected on to $a'\,b'$. Then the line I P meeting C D in K will be the line required.

31. (Fig. 2, Plate VII.).—To find the angle of inclination between two given lines A B, C D. Let the two given lines intersect in I. The horizontal projections i, i' of this point must lie on the perpendicular $i\,i'$ to the ground

line according to the definition of the projection of a point; if this is not the case, we know that the lines do not meet in space.

Find the plane containing the two lines A B, C D. The horizontal trace of this plane is found, as in Art. 25, by finding the horizontal traces a, c of these lines. Now, in order to find the real magnitude of the angle of which the angle $a\,i\,c$ is the horizontal projection, it will be necessary to *construct* the triangle A I C (by Art. 26) by turning it about its horizontal traces $a\,c$ until (I) falls into the paper or horizontal plane. (I) will in this revolution describe a vertical circle, the horizontal projection being the line S $i\,m$ drawn perpendicular to $a\,c$, and the radius being the distance of the point I from $a\,c$; this distance is found as in Art. 26, by setting the height $i\,l$ along $a\,c$ from m to l', and then the distance $l'\,i$ set out from m will give the position S of the point I when turned down. The points a, c will not move during this rotation, as they lie on the axis of rotation; therefore, the lines $a\,i, c\,i$ will assume the position a S, c S, and the angle a S c will be the angle required.

In future we shall not go through the description of the construction necessary to find the real form of any figure, but shall simply state that the figure is to be '*constructed*' about a given line, leaving the student to perform the operation, as in Art. 26 and as above.

32. To find the angle of inclination of a given line and plane.

Def.—The inclination of a straight line to a plane is the acute angle contained by that straight line, and another drawn from the point at which the first line meets the plane to the point at which a perpendicular to the plane drawn from any point in the first line above the plane meets the same plane.

From the above definition we see that the angle between the given line and the perpendicular from any point on the line on to the plane is the complement of the angle required. Hence the following construction:—

(Fig. 3, Plate VII.).—Let A B be the given line and L M N the given plane. Take any point P on the line A B, and from P draw a perpendicular on the plane; by Art. 28 we know that the projections of this perpendicular will be perpendicular respectively to the traces of the plane. All that remains to be done, therefore, is to determine by the previous Art. the angle of inclination between the given line and perpendicular, and the complement of the *acute* angle formed between them will be the angle required.

33. To draw the projections of a line inclined to the horizontal plane at an $\angle\ \alpha°$, and to the vertical plane of projection at an $\angle\ \beta°$.

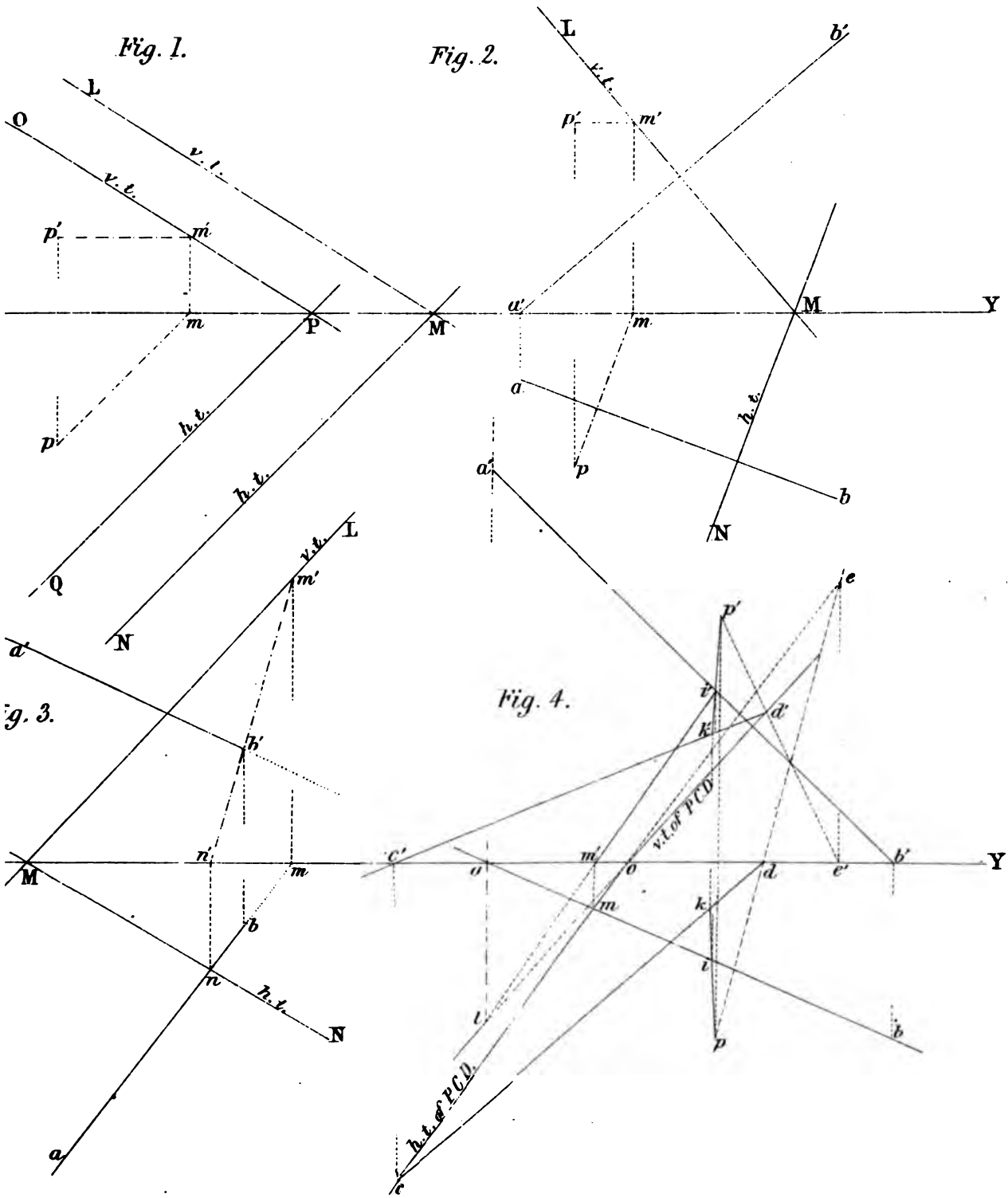


Fig. 1.
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Fig. 3.
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Fig. 1.

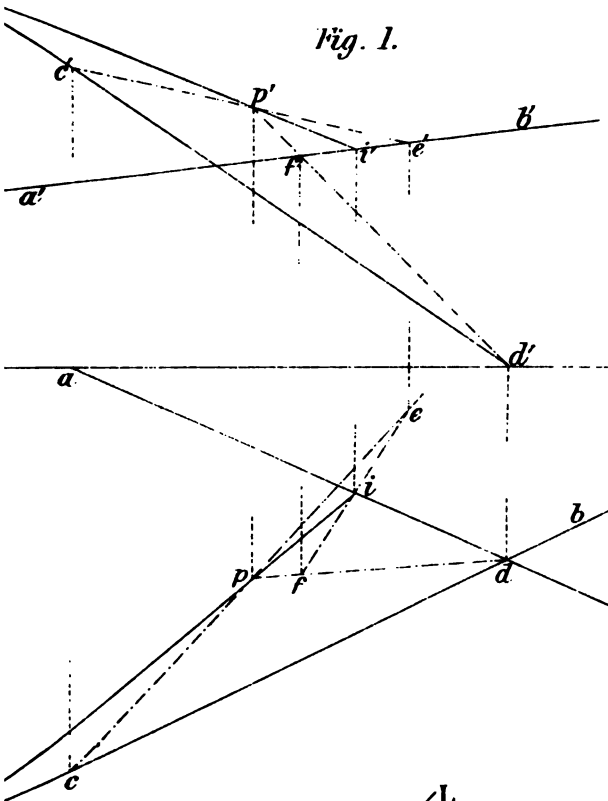


Fig. 2.

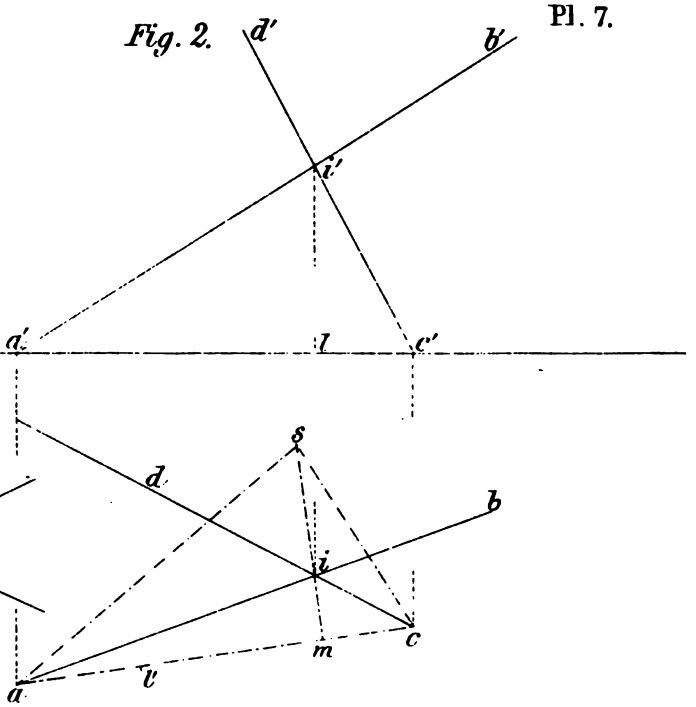
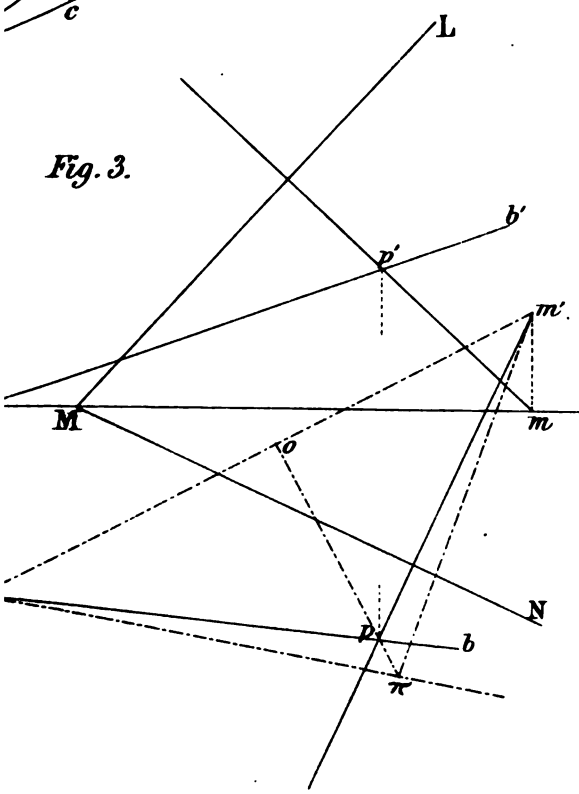


Fig. 3.



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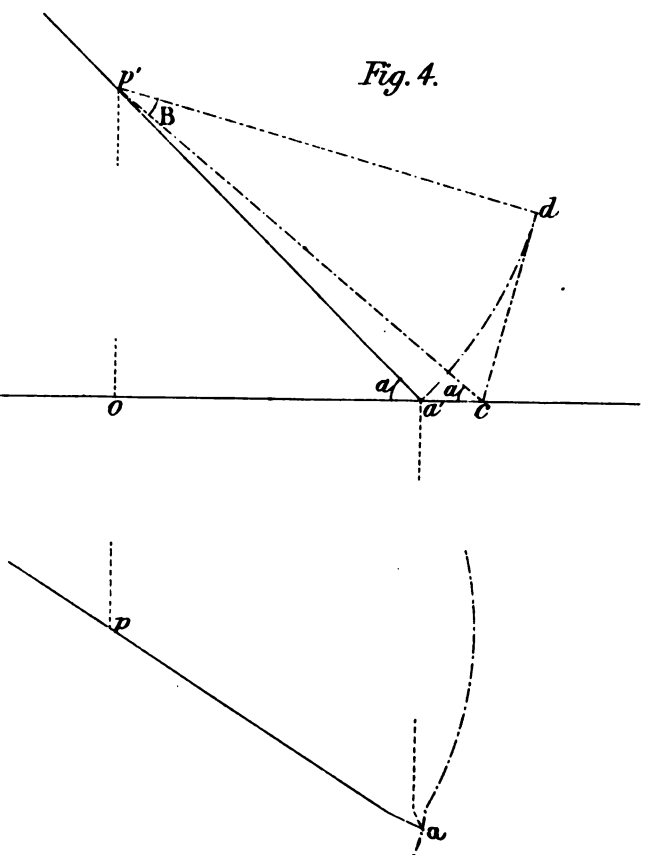


Let p, p' be the projections of a point on the line (Fig. 4, Plate VII.); draw $p'\,c$ inclined to the ground line at the given angle $\alpha°$. Then with p as centre and $o\,c$ as radius, describe a circle. Supposing this to be the plan of a cone formed by the revolution of P c about the vertical through P, this cone will be the locus of all the lines that can be drawn through P inclined at an $\angle\ \alpha$. Again, make the angle $c\,p'\,d = \angle.\ \beta°$, and drop the perpendicular $c\,d'$ on to $p'\,d'$. Then $p'\,d'$ will be the length of the vertical projection of the line required. Therefore, make $p'\,a' = p'\,d'$, and draw $a'\,a$ at right angles to the ground line to cut the plane of base of cone in a, and join $p\,a$; then $p\,a$, $p'\,a'$ will be the projections of the line required.

34. To find the shortest distance between two lines in space which do not meet.

This distance is the line which is at right angles to both the given lines. The construction for finding this line will be best understood by the perspective view in Fig. 2, Plate VIII.

Let A B, C D be the two given lines. Draw a plane through one of the lines A B, parallel to the other line C D. This can always be done by taking any point B in A B, and drawing B L parallel to C D, and the plane determined as containing A B, B L will be the plane required.

Next, take any point P in C D and drop a perpendicular from P on to the plane containing A B, B L (Art. 28), and find the point of intersection I (Art. 29). Through I draw I M parallel to C D, meeting A B in M. Through M draw M N parallel to P I or perpendicular to the plane; it will necessarily meet C D, and be at right angles to it. M N is therefore the distance required. That this is truly the shortest distance between the two lines will be perceived at once by imagining C D to be the axis of a cylinder touching the plane containing A B and B L along the line M I. Now it is evident that M N, the radius of the cylinder, is shorter than any other line joining A B and C D, because any other line drawn from C D to A B would meet A B in a point outside the surface of the cylinder, and will therefore exceed the radius M N.

The projections are shown in Fig. 1, Plate VIII., where the same letters are kept as in Fig. 2, but as the construction required involves nothing that has not already been explained in previous Articles we will leave the working of it out to the student.

35. Through a given point to draw a line parallel to a given plane and in-

clined at a given angle to the horizontal plane (Fig. 3, Plate VIII.)—Let P be the given point, L M N the given plane, α the angle of inclination of the given line.

The first thing to be determined to solve this problem is to find the projections of a line which shall lie in the plane L M N and be inclined at the given angle.

Let a' be the vertical trace of such a line. This must be some point in the vertical trace L M of the plane as the line lies in the plane. Now suppose the line to revolve about its perpendicular projector $a\,a'$ until it become parallel, or, in this case, lies in the vertical co-ordinate plane. Then the inclination of the line with the ground line will express the given angle. Therefore, through a' draw a' B, making the angle a' B a equal to α. The plan of the cone described by the revolving line a' B is the circle described with a as centre and a B as radius; and this circle must contain the horizontal trace of the line; but the horizontal trace of the plane must also contain this point; therefore b is the horizontal trace of the line, and $a\,b$ is therefore the horizontal projection, and $a'\,b'$, the vertical projection, may now readily be deduced from it by projecting the point b into the ground line.

All that remains to be done is to draw through the point P a line parallel to A B; and as the projections of parallel lines are themselves parallel, this can now be done.

36. Two lines which meet are inclined to each other at a given angle, and are also respectively inclined to the horizontal plane at given angles. To draw the horizontal projections of the lines (Fig. 4, Plate VIII.).

First suppose the given lines to lie in the horizontal plane; and let $\angle\, a$ B c be equal to the given angle γ, and α, β their proposed inclinations to the horizontal plane.

Assume any point a in a B and make $\angle$ B $a f = \angle\, \alpha$; draw Bf perpendicular to af; then the line af will be the length of the horizontal projection of a B when inclined at an angle α and B is above a, a distance equal to Bf. With B as centre and Bf as radius describe the circle fg; draw cg a tangent to this circle and inclined to c B equal to the other given angle $\beta°$. Then cg will be the length of the horizontal projection of the line c B when inclined at an $\angle\, \beta°$; and since B is the same distance above c (viz. Bg) as it was taken above a (viz. Bf), it is evident that these two points a and c as two points on the same level; and, therefore, the line joining them may be taken as a horizontal line in the

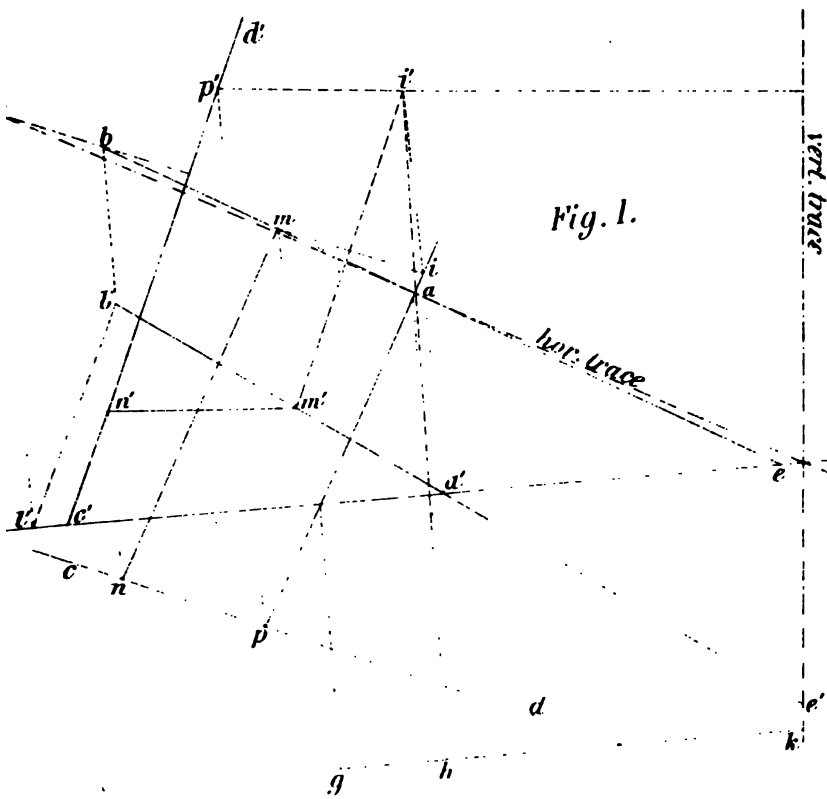


Fig. 1.

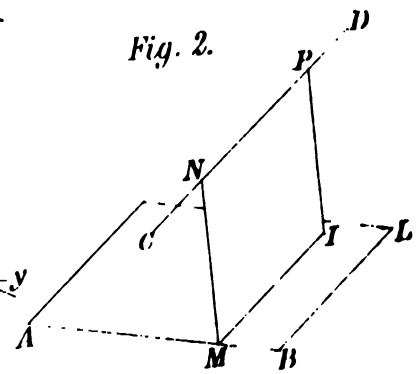


Fig. 2.

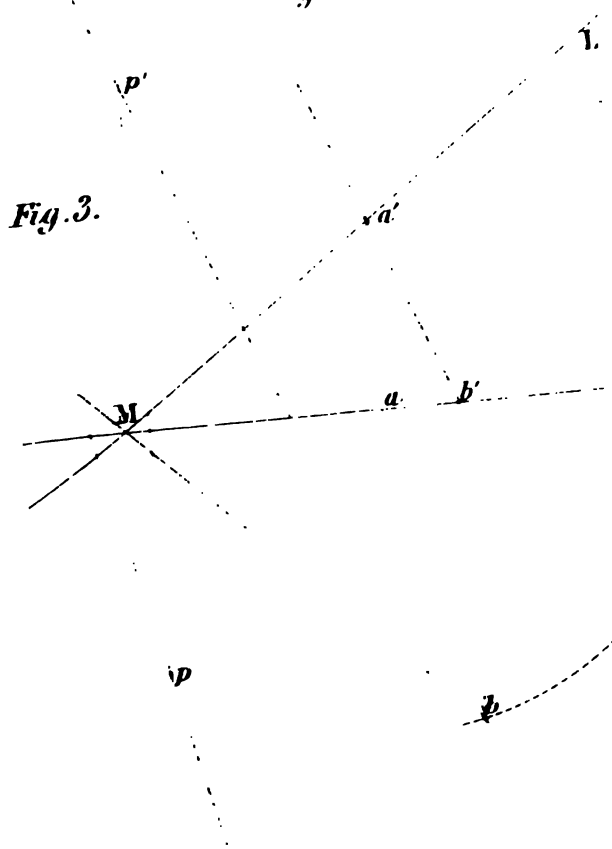


Fig. 3.

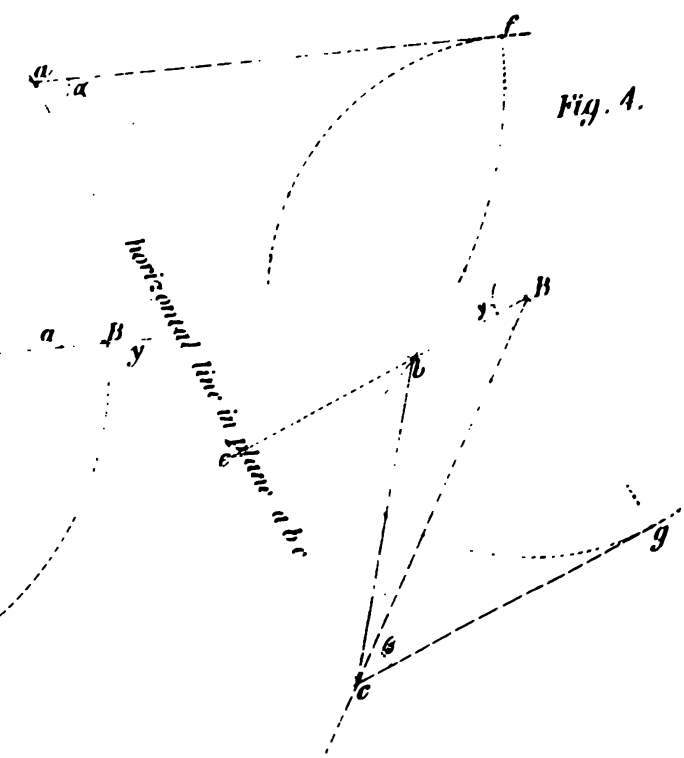


Fig. 4.

plane of the two lines when inclined as proposed. We may now suppose the triangle $a\,B\,c$ to revolve about the line $a\,c$ until B is raised to a height equal to Bf or Bg. Therefore B will travel in the vertical circle projected horizontally in the line Bc perpendicular to the axis of rotation $a\,c$. But the point a will not move; and the length of the horizontal projection of a B is $a\,f$; therefore, with a as centre and $a\,f$ as radius describe the circle $f\,b$ intersecting Bc in b. This point will be the plan of B when raised a height equal to Bf. Similarly, if we take c as centre and $c\,g$ as radius and describe a circle, we shall find it will intersect the arc $f\,b$ on the line Bc, and this will afford a means of verifying the accuracy of the construction.

Lastly, join $b\,a$, $b\,c$, and they will be the horizontal projections of the lines required.

CHAPTER III.

THE FIVE REGULAR SOLIDS.

37. There are only five solids having sides composed of equal regular polygons that can be inscribed in a sphere; these are the Tetrahedron, composed of four equilateral triangles, the Cube composed of six squares, the Octahedron, composed of eight equilateral triangles, the Dodecahedron, composed of twelve pentagons, and the Icosahedron composed of twenty equilateral triangles.

38. To draw the projections of a tetrahedron standing on one face.

Draw the equilateral triangle $a\,b\,c$ (Fig. 1, Plate IX.) and find its centre d, join $d\,a$, $d\,b$, $d\,c$; this will be the plan of the solid.

To find its elevation, all that is required is to find the height of the vertex D. Suppose the vertical plane passing through D c, to revolve about its horizontal trace $d\,c$ until it coincides with the horizontal plane. The point D will then be somewhere on the line $d\,e$ drawn perpendicular to $d\,c$, as this line represents the vertical circle described by D. Now the real length of D c being an edge of the solid is equal to $c\,b$. Therefore with c as centre and $c\,b$ as radius, describe a circle to cut the line $d\,e$ in e, then $d\,e$ will be the height of the vertex.

Set up $d\,e$ from δ to d' and join d' to a', b' and c'.

39. To draw the projection of a cube when resting on one of its faces. (Fig. 2, Plate IX.)

The student will experience no difficulty in projecting this solid, as the height of the top face is equal to the side of the square $a\,b\,c\,d$ on which it rests.

40. To draw the projections of an octahedron when resting on a face $a\,b\,c$. (Fig. 3, Plate IX.)

The projection of the top face $d\,e\,f$ will have the same centre as $a\,b\,c$, but having its angles alternating or placed mid-way between those of $a\,b\,c$. Join

$a\,d$, $d\,b$, $b\,e$, $e\,c$, $c\,f$ and $f\,a$. This is the plan required. To find the elevation, all that is required is to find the distance of the top face D E F from the bottom face $a\,b\,c$. This is done as in case of the tetrahedron by turning the vertical plane $g\,m\,b$ about its horizontal trace $g\,b$ until it coincides with the horizontal plane. Now the real length of $m\,b$ is the line $g\,b$; therefore with b as centre and $b\,g$ as radius, describe a circle to cut the line $m\,h$ drawn at right angles to $g\,b$ in h; then $m\,h$ will be the height required. Draw the line $f'\,d'\,e'$ parallel to the ground line and distant from it equal to $m\,h$, and project the points $f\,d\,e$ into it, and the points $a\,b\,c$ into the ground line, and join the same points in the elevation that are joined in the plan, and the projections of the solid will be completed.

41. To draw the projections of a dodecahedron when resting on one face. (Fig. 4, Plate IX.)

Let the pentagon $a\,b\,c\,d\,e$ be the face on which it rests. Its opposite and top face will have for its projection the equal and concentric pentagon $f\,g\,h\,i\,k$ with its angles alternating. Now suppose the two pentagonal faces having $e\,a$, $e\,d$ for sides respectively to revolve about these lines until they coincide with the horizontal plane. In this position they will be represented by the two regular pentagons $e\,a$ R L E and $e\,d$ U P E. Next let these pentagons simultaneously revolve about the lines $e\,a$, $e\,d$ until the adjoining edges E e of each pentagon coincide. In one case E describes the vertical circle, having for its plan the line E $q\,w$ drawn perpendicular to $a\,e$ produced; and in the other case e E describes the vertical circle E $q\,x$ drawn perpendicular to $d\,e$ produced. Hence, the point q where these two perpendiculars intersect will be the plan of E where the two lines E e coincide. Lastly, by making $w\,f\,l$, $w\,a\,r$, $w\,g\,m$, $w\,b\,s$ and so on, all equal to $w\,e\,q$ and joining the points as in the figure, the plan of the solid will be completed.

The heights of the several points are found in precisely the same way as in the case of tetrahedron and octahedron, by supposing a vertical cutting plane turned down about its horizontal trace, here supposed as $q\,n$, and as no new difficulty will be met with, the student is left to complete the elevation on any assumed ground line.

42. To draw the projections of an icosahedron when resting on a given side $a\,b\,c$. (Fig. 5, Plate IX.)

Having drawn the two equilateral triangles $a\,b\,c$, $d\,e\,f$ as representing the

top and bottom faces, observe that the figure $a\,b\,k\,d\,l$ is a regular pentagon, being in fact the base of a pentagonal pyramid formed by five equilateral triangles meeting in a point. Therefore, suppose the two pentagons having $a\,b$, $a\,c$ as bases and d, f as vertices to be turned down into the horizontal plane in the same manner as in the former example of the dodecahedron; the point k will be found in precisely the same method and the plan may now be completed by making $w\,d\,g$, $w\,b\,l$, &c., all equal to $w\,a\,k$, and an elevation constructed on any given ground line by taking the heights from the vertical cutting plane $i\,w\,l$ as in the preceding examples.

43. The two projections shown in Plate X. are intended as studies for the student in accurate drawing. Fig. 1 shows the projections of a dodecahedron having its axis vertical; this is shown by the blue lines, and the method of construction is explained in Art. 44. The faces of each pentagon are then produced, forming twelve pentagonal based right pyramids, one on each face. The vertices of these pyramids are then joined in threes, forming twenty equilateral triangles, or an icosahedron; this solid is shown in the figure by means of red lines.

Fig. 2. The blue lines in this figure show the projections of an icosahedron, having one face horizontal. The sides are produced so as to form twenty pyramids; the vertices of which, taken five together, being joined, form twelve pentagons or a dodecahedron; this solid is shown in the figure by red lines.

44. To draw the projections of a dodecahedron, having its axis vertical. (Fig. 1, Plate XI.)

In this position, the solid will rest on one of its solid angles; and an horizontal plane, cutting the edges of this solid angle at their extremities, will form an equilateral triangle $a\,b\,c$, of which the side is equal to the true length of the diagonal of the pentagon which is given. Join $o\,a$, $o\,b$, $o\,c$, these will represent the plan of the lines forming the inferior solid angle at o. Now, suppose the pentagon of which $a\,b$ is the diagonal to revolve about this horizontal line until it become itself horizontal, and therefore represented in plan in its true form, as $O\,b\,D\,E\,a$. Take the line $m\,l\,O$, bisecting D E in m as a ground line, the same level as $a\,b$. Next, suppose the pentagon to revolve back again into its original position, and make an elevation of the plan on the ground line $m\,l\,o$. It is evident that the points $a\,b$, which are projected vertically in l, will not move as

THE FIVE REGULAR SOLIDS.

Fig. 1.

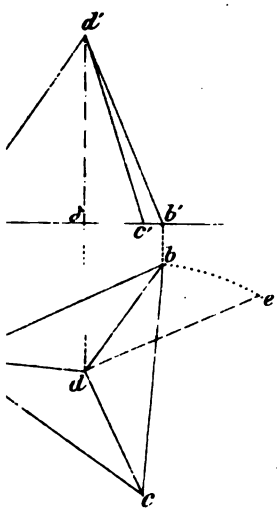


Fig. 2.

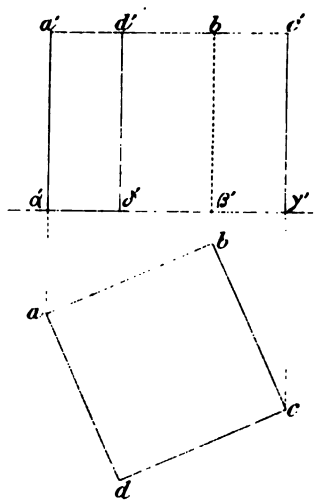
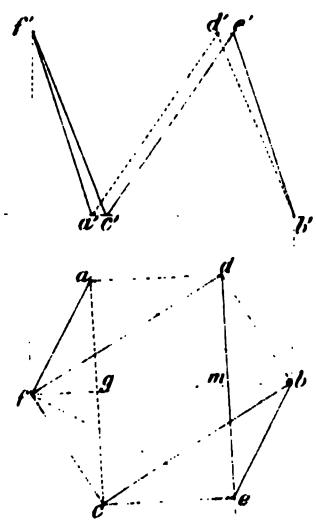


Fig. 3.



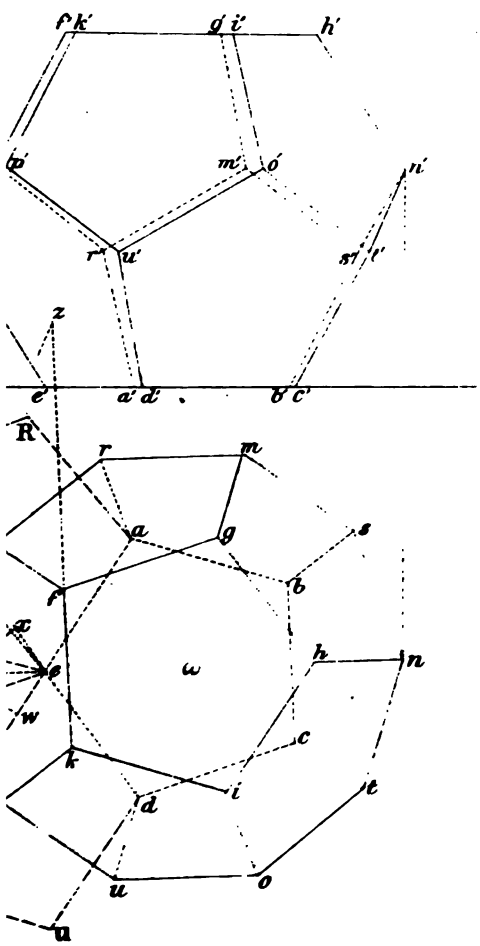
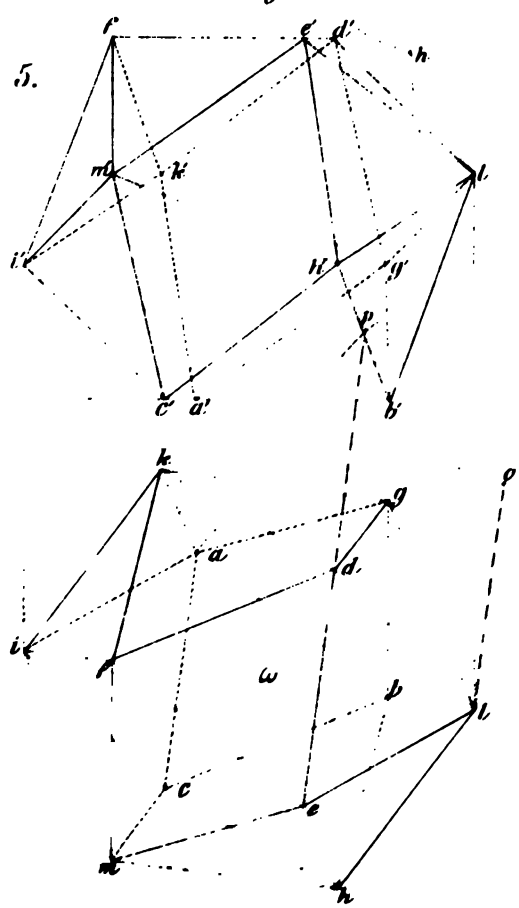
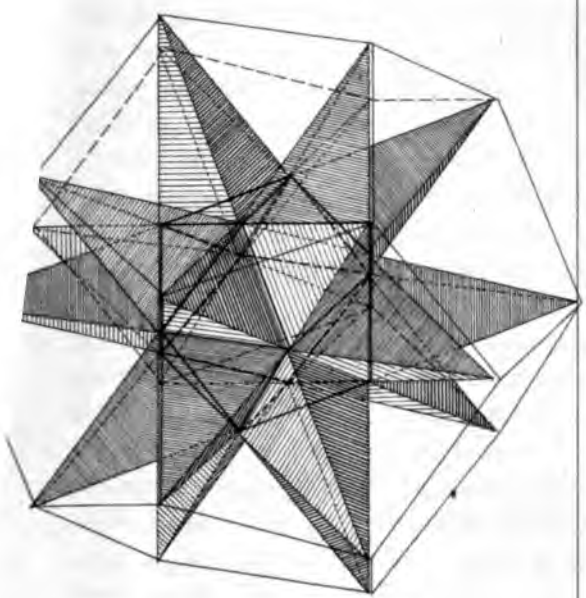
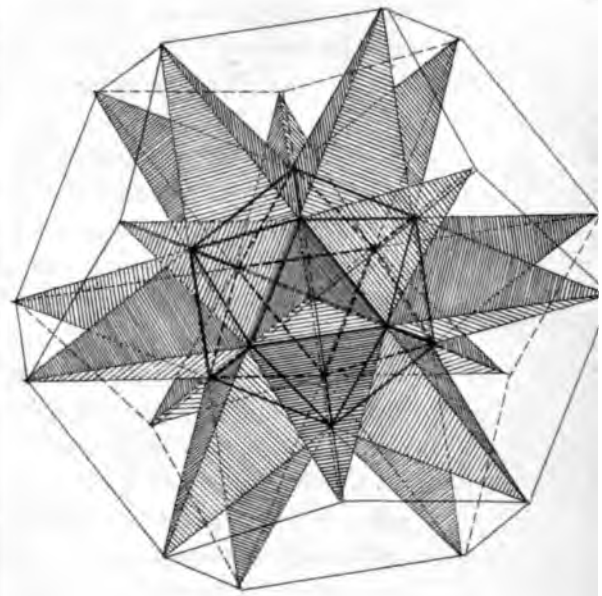
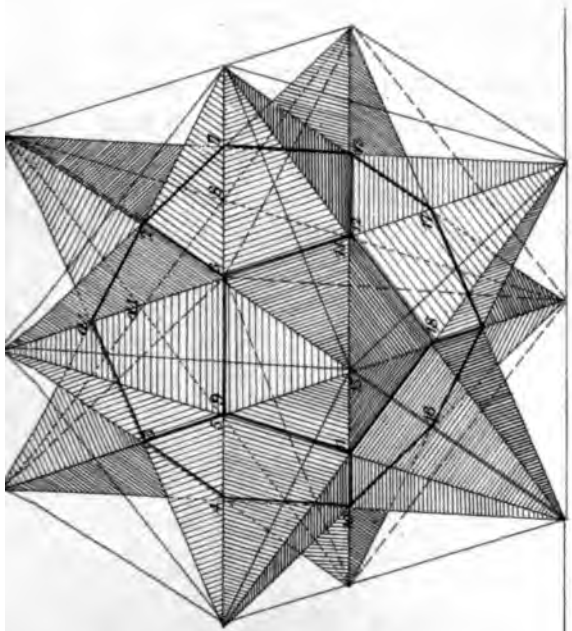


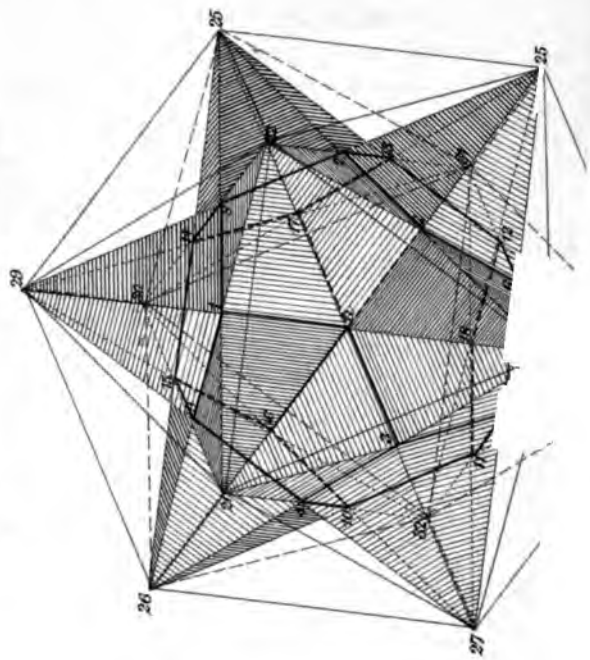
Fig. 5.







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they lie on the axis of rotation ; but the point O will resume its original position at *o*, being the centre of the equilateral triangle *a b c*; to find the vertical projection of this point, draw *o o'* perpendicular to *m l o*, and with centre *l* and radius *l* O describe the arc O *o*, cutting *o o'*, and *o'* the point required.

Join *o' l* and produce it; this line will represent the vertical projection of the pentagon, and its inclination to the line *l o* will represent the inclination of the pentagon to the horizontal plane. During the revolution, the points D and E will describe vertical circles about the axis *a b*; therefore their plans will be the straight lines D *d*, E *e* drawn perpendicular to *a b*. The vertical projection of these circles is found by taking *l* as centre and radius equal to *l m*, describing the arc *l* M, to cut the line *o' l* produced in M; and lastly, by dropping a perpendicular from M to *m l o*, to meet D *d* and E *e* in the points *d* and *e*, and joining *d e*, *e a*, *a o*, *o b*, *b d*, we have the horizontal projection of one of the pentagons forming the inferior solid angle, and hence the other two may be easily found. The three pentagons forming the top solid angle will have the same point *o* as plan, only they will range mediately with respect to those of the lower angle. Having completed these figures, the lines joining the sides may now be filled in, and the plan will be finished.

In constructing a vertical projection on any given ground line *x y*, it is only necessary to transfer the heights of the several points as follows:—

The height of points A B and C above the ground is = *o o'*. The height of the six points the same level as D E is = *o o'* + M K; the height of the six points the same level as F G is = *o o'* + *q p*; this last distance is found by supposing the vertical plane *b p* to be turned down about the horizontal line *b p* until it is horizontal or parallel to the paper; the section formed will be the right-angled triangle *b p q*, found by taking *b* as centre, and with radius equal to the diagonal of pentagon, viz: O *m* describing an arc cutting *f g* produced in *q*; then *p q* will be the height of F G above B. Of course the points R S T will be as much above the level of F G . . . as A B C are below D E . . . Similarly for the top of the solid O.

CHAPTER IV.

EXAMPLES AND PROBLEMS ILLUSTRATING THE PRINCIPLES OF CONSTRUCTION CONTAINED IN THE PRECEDING CHAPTERS.

45. To draw the projections of a cube (edge = 1″), having given the inclination (50°) of one of its faces and of one of the edges of said faces (30°).

(Fig. 2, Plate XI.) Let L O M be the given plane assumed at right angles to the vertical plane of projection, and therefore having its horizontal trace at right angles to the ground line and its vertical trace inclined to the ground line the given angle. Assume a point A (a a') lying in this plane, as one of the angles of the cube. The problem is to find the plan of a line through A lying in the plane L O M and inclined to the horizontal plane at an angle of 30°. To do this, as shown in Art. 35, we draw $a'f$ inclined to xy at the given angle 30°; and with a as centre and radius equal to ef describe an arc cutting the horizontal trace in g; then ga will be the plan of the line required. Next let the plane L O M revolve about its horizontal trace O M until it coincides with the horizontal plane; the point A will describe a vertical circle, projected horizontally in the line a A ^{r} to O M and vertically by the circle a' A′ described as the centre O. By drawing A′ A at right angles to X Y, A will be the position of the point when the plane is turned down. But the point g being on the axis of rotation does not move, therefore A g will be the position of an edge of the cube when the plane coincides with the paper. Made A D on A g = 1″ and on A D construct the square A B C D, and then suppose the plane to revolve back again into its first position carrying with it the points A B C D. A, of course, will resume its first position at a for its horizontal projection, and b may be found in like manner by drawing B B′ at right angles to X Y, and with O as centre and O B′ radius describe the circle B′ b' cutting L O in b', and lastly drawing b b $\perp^{r}$ X Y cutting B b drawn at right angles to the horizontal trace in b; and similarly with the other two points C and D.

Another and more elegant method is to produce B A to meet O M in k, join k a and produce it to meet B b in b the point required, and in like manner

Fig. 1.

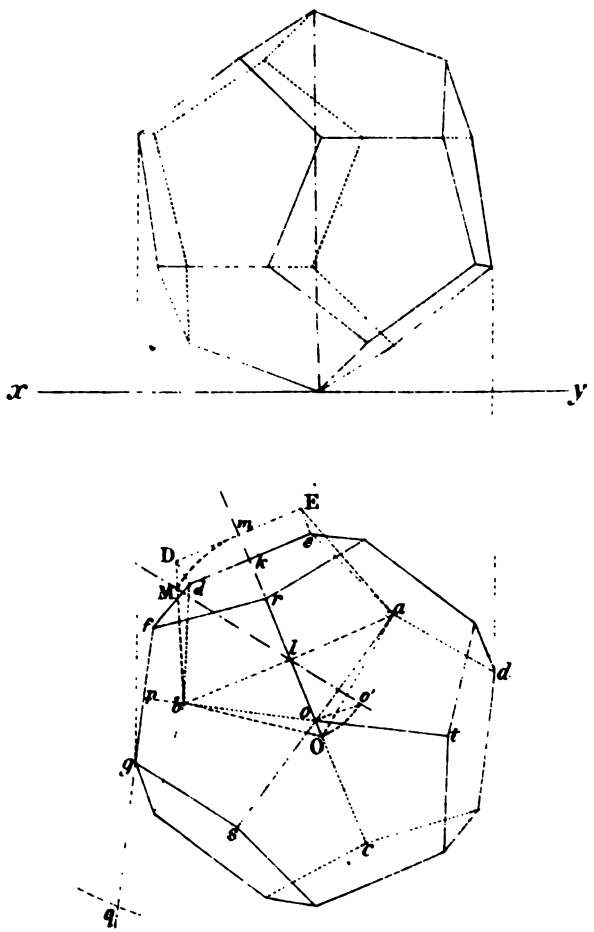
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Fig. 2.

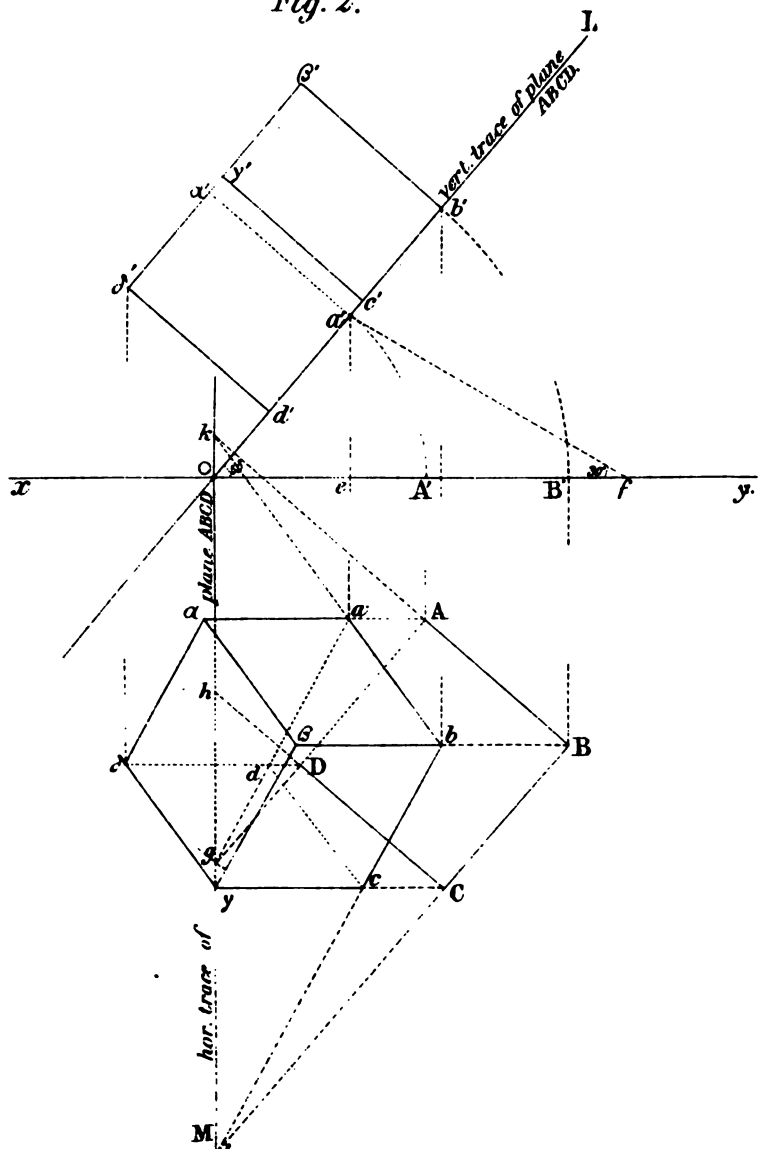
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Fig. 1.

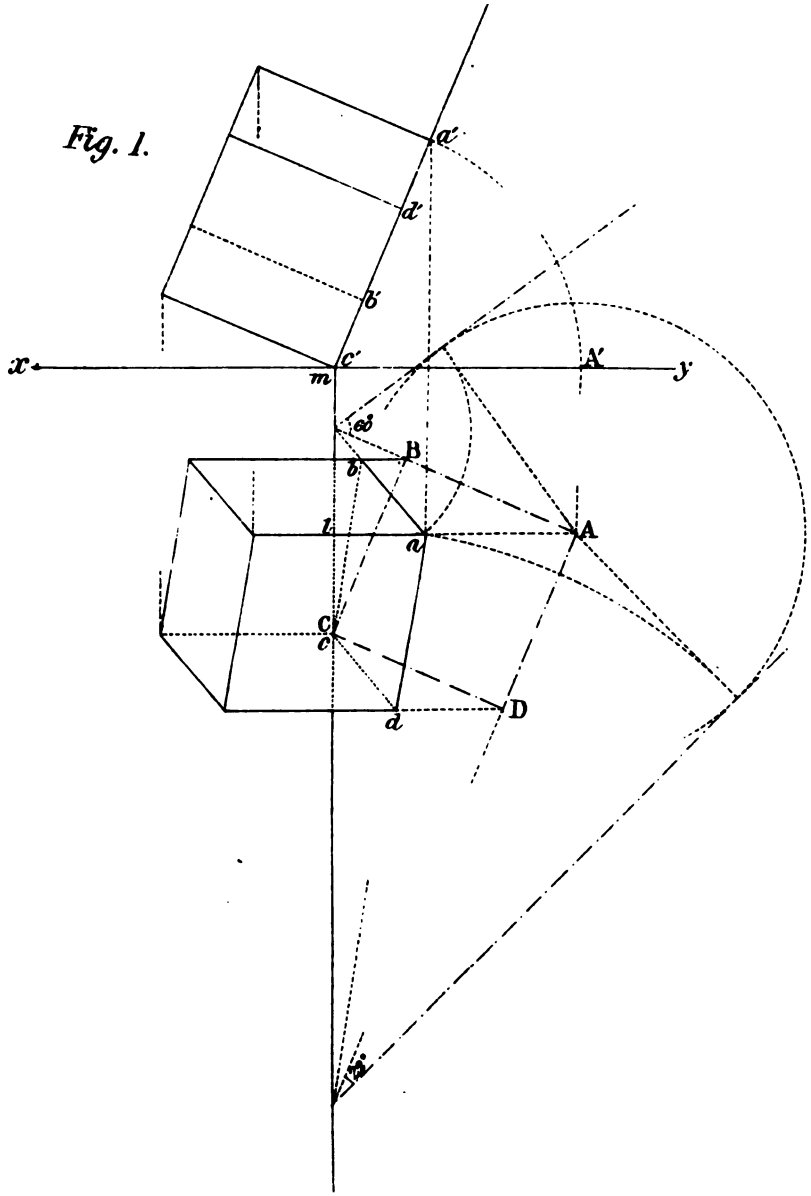
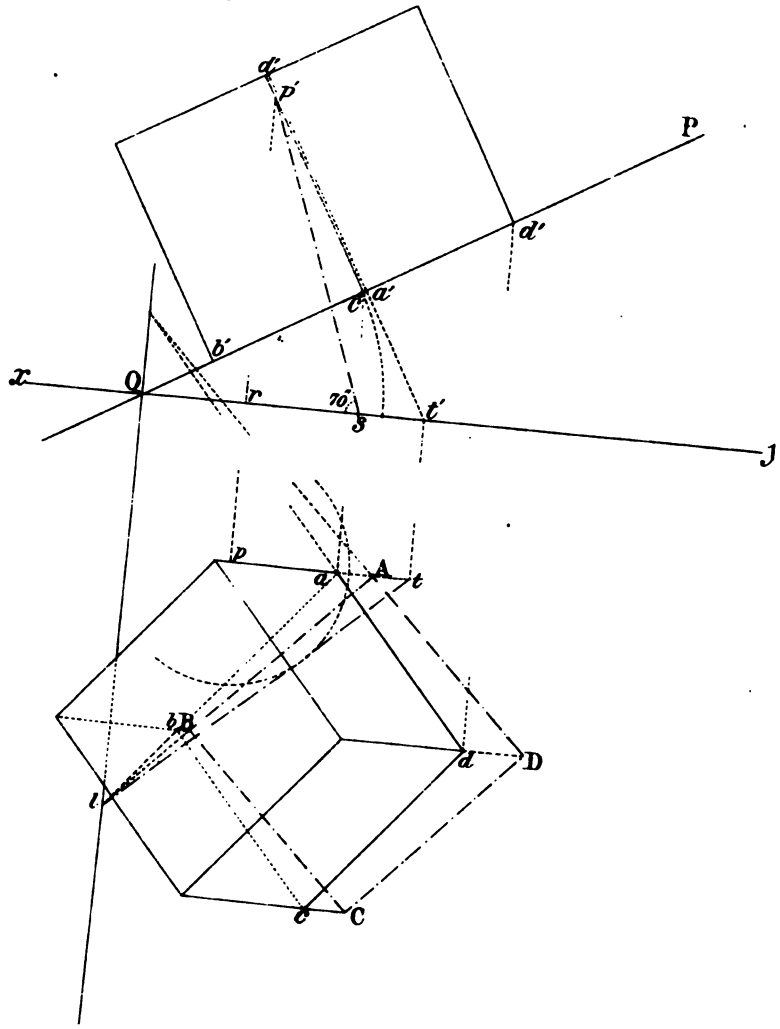


Fig. 2.



for the other sides of the square as shown in the figure. Next project the points *b c d* into the vertical trace L O by lines at right angles to X Y. Through A B C D draw lines at right angles to the plane L O M. This is done by drawing lines at right angles to their respective traces (see Art. 28). These perpendiculars will be projected vertically in their true length as they are parallel to the vertical plane of projection; therefore draw $b'\beta'$, $c'\gamma'$, $a'a'$, $d'\delta'$ each equal to 1″, and join $\delta\beta'$ and the vertical projection of the cube will be completed. The plan may now be finished by drawing the lines $\beta'\beta$, $\gamma'\gamma$, $a'a$, $\delta'\delta$, at right angles to X Y to meet $a\,a$, $b\,\beta$, $c\,\gamma$, $d\,\delta$ in a, β, γ, δ respectively. Lastly, join $a\,\beta$, $\beta\,\gamma$, $\gamma\,\delta$, $\delta\,a$, and the plan is completed. The three lower edges $a\,a$, $a\,c$, $a\,\delta$ in plan will be dotted, and in elevation, the vertical projection $a'\,a'$ being the edge nearest the vertical plane of projection and therefore behind the others.

46. To draw the projections of a cube, having given the inclinations (60° and 22°) of two contiguous edges to the horizontal plane, and the length of an edge 1″.

(Fig. 2, Plate XII.) First determine any horizontal line in the plane containing the two edges A B, A D inclined as given by the construction explained in Art. 36. Having thus determined the plane *a b c d* of this face, assume any ground line X Y at right angles to the horizontal line of plane as a line of level; with *m* as centre and radius equal to *l* A describe the arc A′ *a*′, draw *a a*′ at right angles to X Y cutting said arc in *a*′; join *m a*′, this line is the vertical trace of the plane A B C D, into which the points *a*, *b*, *c*, *d* are to be projected and the remainder of the cube drawn as in the previous example.

47. To draw the projections of a cube, having given the inclinations of two adjoining faces 30° and 70° to the horizontal plane. Edge of cube = 1·25″.

Draw the plane P Q R inclined at 30°, assumed for simplicity to be perpendicular to the vertical plane of projection. Assume any point (a, a') in this plane, and draw the line $(a\,t, a'\,t')$ at right angles to the plane P Q R. Find its horizontal trace (t, t'). Since the contiguous plane inclined at 70° will contain this perpendicular, its horizontal trace must pass through *t*. Again, take any point in the perpendicular $(a\,t, a'\,t')$, in the example the point P is taken, and make it the vertex of a right cone having its generating line inclined to the horizontal plane at an ∠ 70°. The plan of this cone is the circle described with *a* as centre and radius *r s*. Now the plane which contains (t, t') and touches

this cone will evidently be inclined at 70°, and as it contains the line A T, will be the contiguous face of the cube required. The horizontal trace of this plane is found by drawing the tangent $t\,l$, cutting Q R in l. Join $l\,a$, this represents the plan of the line of intersection of the two planes and therefore the direction of an edge of the cube. Construct the point (a, a') in A and join A l; set up A B = 1·25″, and complete the projections as in the two preceding examples.

48. To draw the plan of a cube, having given the directions of the three edges forming a solid angle and the projected length of one of said edges.

The solution of this problem depends upon the theorem enunciated in Art. 28, viz. that when a line is perpendicular to a plane the projections of the line are respectively perpendicular to the traces of the plane.

(Fig. 1, Plate XIII.) Let $a\,l$, $a\,m$, $a\,n$ be the horizontal projections of the upper solid angle of the cube; and let $a\,b$ be the projected length of one of the edges. Since A N is at right angles to the plane containing A L and A M, $a\,n$ is at right angles to its horizontal trace, and, therefore, to the projection of any horizontal line in the plane. Draw any line $l\,m$ at right angles to $a\,n$ cutting $a\,l$, $a\,m$, in l and m; $l\,m$ will represent a horizontal line. Suppose the plane l A m to revolve about the line $l\,m$ until it becomes horizontal; the point A will evidently be somewhere in $a\,n$, which is the plan of the vertical circle it describes. Again, since the angle l A m is a right angle, it will be somewhere in the circumference of the semicircle l A m described on $l\,m$; therefore the point A will be at the intersection of $a\,n$ with the semi-circle. Join A l, A m and through b draw b B perpendicular to $l\,n$ cutting A l in B, make A D equal to A B and draw D d perpendicular to $l\,n$ cutting $a\,m$ in d, complete the parallelogram $a\,b\,c\,d$, it is the projection of one face. Draw X Y perpendicular to $l\,m$, as a line of level $l\,m$, and draw $a\,a'$ at right angles to X Y. With centre o where X Y cuts $l\,m$ and radius o A′ describe the circle A′ a' cutting $a\,a'$ in a'; join $a'\,o$; it is the vertical trace of plane A B C D. Through a' draw $a'\,a'$ perpendicular to $o\,a'$ and equal to A B; and make $a'\,a$ perpendicular to X Y cutting $a\,n$ in a. Make $b\,\beta$, c γ, $d\,\delta$ all equal and parallel to $a\,a$ and join $a\,\beta$, $\beta\,\gamma$, $\gamma\,\delta$ and $\delta\,a$ and the projection will be completed.

49. To draw the plan of a cube having three corners of one of its faces 1^{in}, $1{\cdot}75^{in}$ and 2^{in} respectively above the horizontal plane. Edge of cube = $1{\cdot}5^{in}$.

Fig. 1.

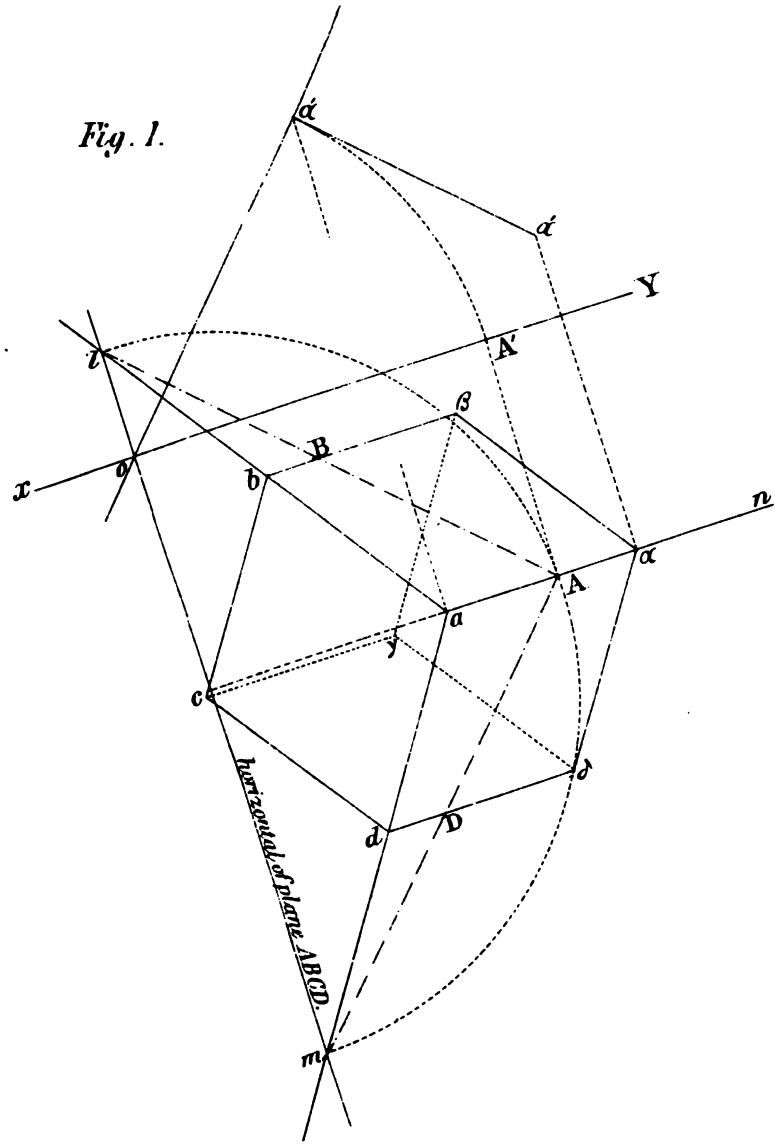

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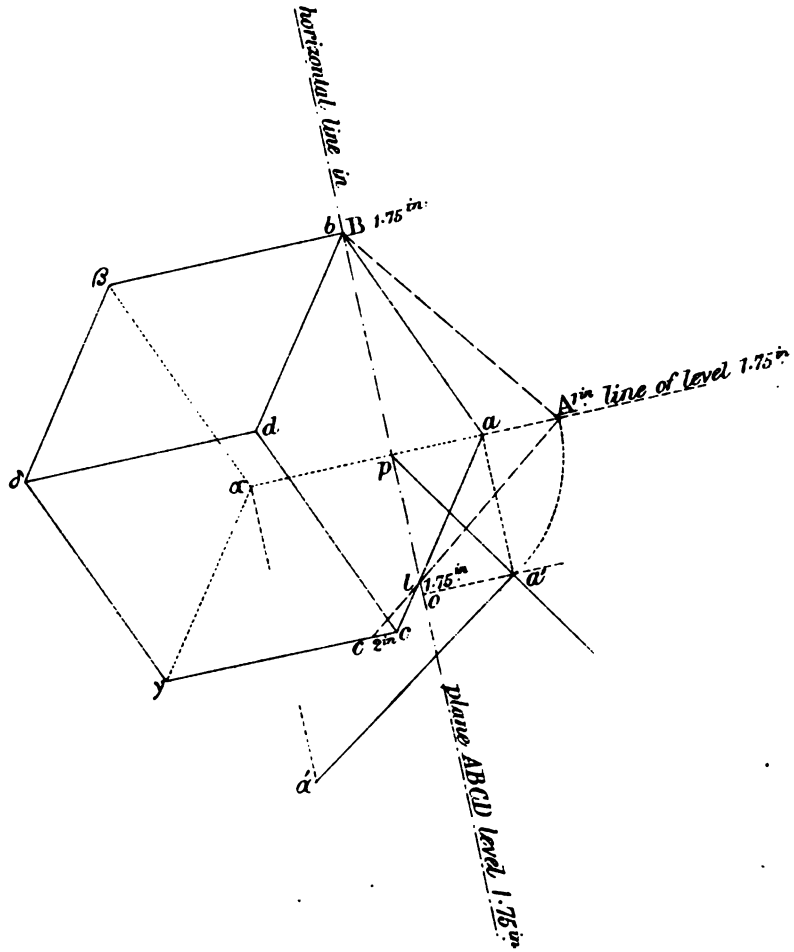
Fig. 2.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ά
plane ABCD level 1·75 in.

(Fig. 2, Plate XIII.) First, suppose the face B A C to be horizontal. Let the index of B be $1{\cdot}75^{in}$ A 1^{in} and C 2^{in}. In A C take A $l = \frac{3}{4}$ A C, then the index of l is $1{\cdot}75^{in}$. Join B l; this line may be considered a horizontal line in the plane A B C D about which to revolve the plane. Draw A p perpendicular to B l, this is the plan of the vertical circle described by A. Also considering this line as a line of level 1·75, the vertical projection of this circle is found by taking p as centre and p A as radius and describing the circle A a'; and since the point A in order to be 1 inch above the horizontal plane must be ·75 of an inch below the line of level $1{\cdot}75^{in}$, draw the line $o\ a'$ parallel to p A and ·75 of an inch below it, to cut the circle A a' in a', and drop the perpendicular $a'\ a$ not o p A; then a is the horizontal projection of the point A and a B of the edge A B. The point l where A C cuts B l remains stationary during the revolution, therefore join $a\ l$ and produce it to cut C c drawn at right angles to B l in c. Complete the parallelogram $a\ c\ d$ B, it will be the plan of one face of the cube. Join $p\ a'$, the angle $a'\ p$ A expresses the inclination of this face to the horizontal plane; draw $a'\ a'$ at right angles to $p\ a'$ and equal to A B or A C; drop the perpendicular $a'\ a$ on the line $a\ a$ and complete the projection of the cube as in the previous examples.

50. Draw the plan of a cube having its axis vertical; edge = 2^{in}.

In this position the three faces forming the superior solid angle and the three forming the inferior solid angle which terminate the vertical axis, will be equally inclined to the horizontal plane, and therefore the edges bounding the cube will be equal inclined.

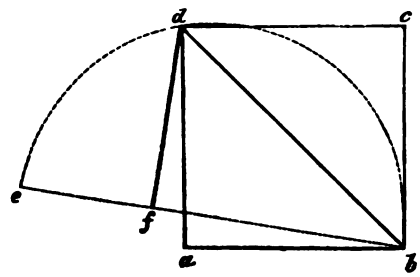
(Fig. 1, Plate XIV.) Suppose A B C D to be the position of one of the superior faces forming the solid angle A when horizontal, and the same level as A, the summit of the vertical axis. Through A draw the line l A m parallel to D B; and suppose the plane A B C D to be turned down about this line until it make the slope required. The points D, C, and B will revolve in the vertical circles, projected horizontally by the straight lines Dd, Cc, Bb drawn $\perp^r$ to l A m. But since the three edges A a, A d, A b, which form the solid angle at A, are equal inclined and symmetrically situate with respect to the vertical axis, their projections will make equal angles with each other; and therefore the $\angle\ d$ A b = $\angle\ b$ A a = $\angle\ a$A d = 120°. Draw the angles C A b, C A d equal to 60°, and let A d, A b cut D d, B b in d and b respectively. Make A a equal to A d or A b. The edges forming the inferior solid angle will be equally projected with the angles alternating, as A c, Aδ, A β. Join $b\ c$, $c\ d$,

$d\delta$, δa, $a\beta$, βb, and the plan will be completed. This projection is called the isometric projection of the cube, because the edges of each of the three faces forming the solid right angle are projected equally.

51. In a given sphere to inscribe a cube, having given the position of an angle and the direction of an edge passing through it.

(Fig. 2, Plate XIV.) Let M N S be the projection of given sphere, a the angle of cube, and $a\,b$ the direction of an edge. It is first necessary to determine the real length of an edge of the cube to be inscribed in the sphere; or, since the diagonal of the cube is equal to the diameter of the sphere, to determine the relation between the edge and diagonal of a cube, so that having given the latter we may deduce the former.

Let $a\,b\,c\,d$ be the plan of a cube resting on the paper, and suppose $d\,e\,b$ to



be the half section formed by the vertical plane $d\,b$ when turned down about $d\,b$ into the paper. Then $e\,b$ is the diagonal of cube.

If $ab = a$ (from Euc. b. i. p. 47) $db = a\sqrt{2}$
and $be = a\sqrt{3}$,
draw $df \perp^r eb$. Then from similar triangles
$ef : ed :: ed : eb$,
or $ef : a :: a : a\sqrt{3}$

$$\therefore ef = \frac{a}{\sqrt{3}} = \frac{a\sqrt{3}}{3} = \frac{eb}{3}.$$

Hence the following construction for determining the side, having given the diagonal of the cube $e\,b$.

Take $ef = \frac{eb}{3}$ and raise the $\perp^r fd$ to cut the semicircle described on $e\,b$ in d; $e\,d$ is the length required.

Fig. 1.

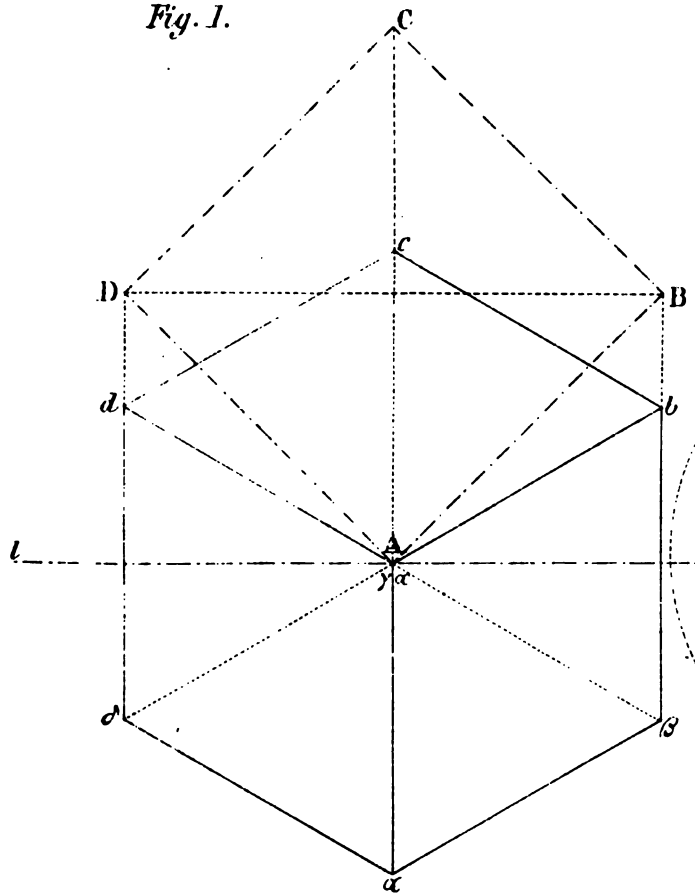
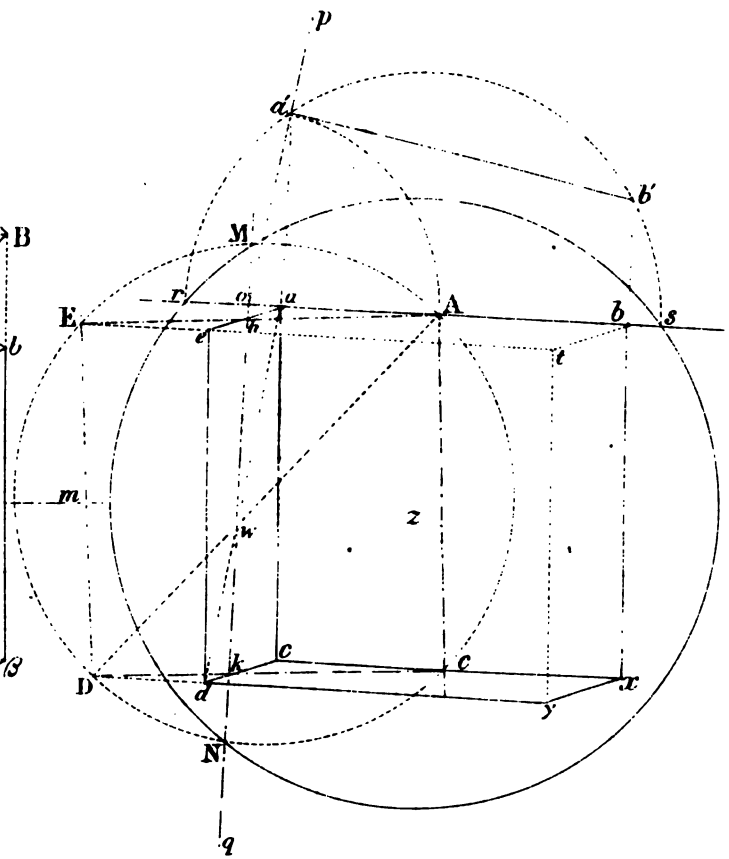
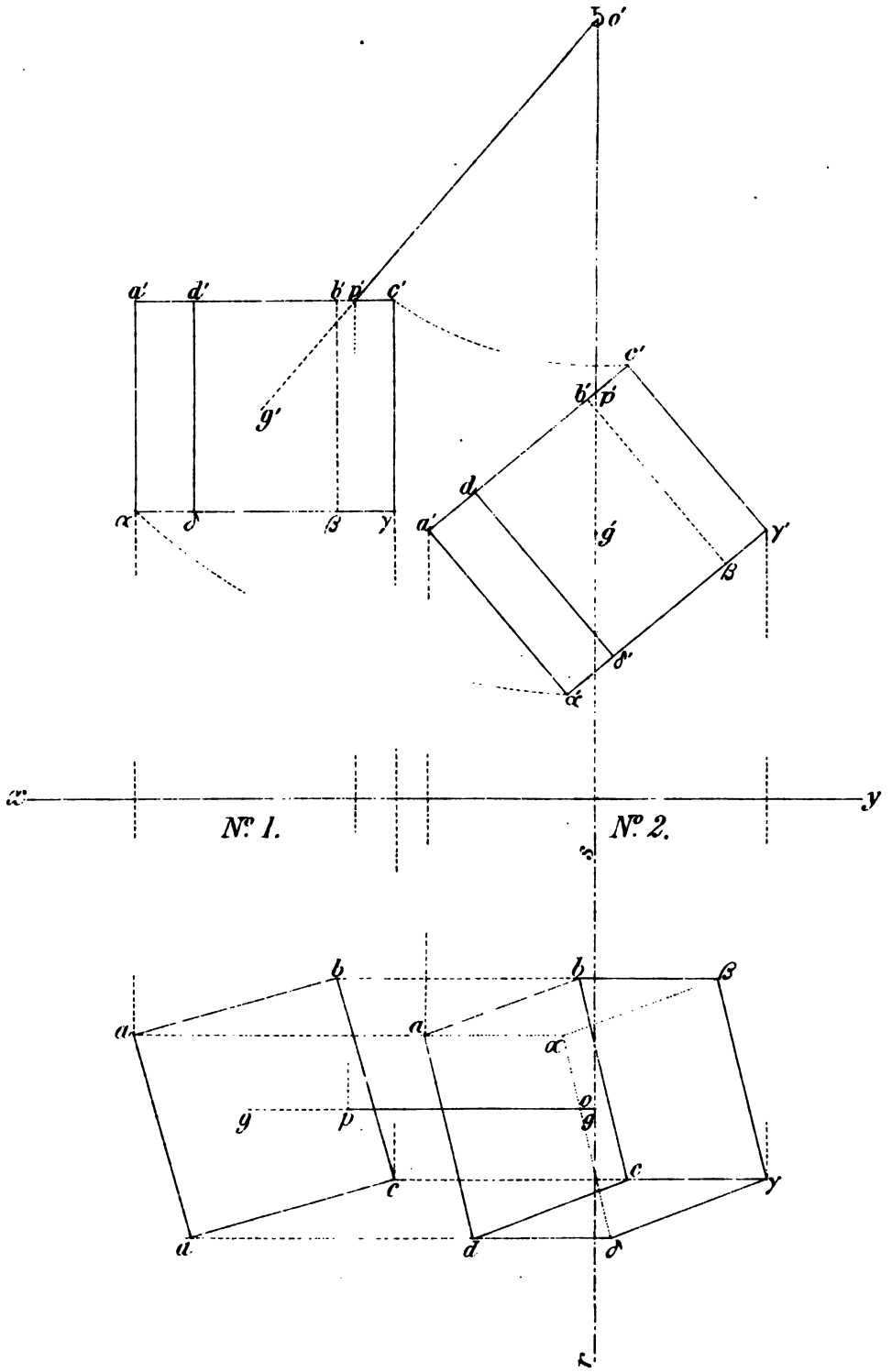


Fig. 2.




o'
a' d' b' p' c'
g'
α δ β γ
c'
b' p'
d
g'
a' γ'
β
δ'
α'
x
y
N° 1.
N° 2.
z
b b β
a a α
g p o
θ
c c γ
d d δ
z

This being found, suppose the line $a\,b$ to be a line of level the same as the centre of sphere, cutting the great horizontal circle of the sphere $m\,n\,s$ in $r\,s$; on $r\,s$ describe the semicircle $r\,a'\,s$, it will be the vertical projection of the vertical circle $t\,a\,s$ which contains the edge of cube. Draw $a\,a' \perp^r r\,s$ to cut circle in a'. With a' as centre and radius equal to edge of cube found as above, describe an arc cutting circle in b', draw $b'\,b \perp^r r\,s$ cutting $a\,b$ in b; then $a\,b$, $a'\,b'$ are the two projections of edge A B. Next draw the plane of face perpendicular to A B through A. The vertical trace (Art. 28) is found by drawing $p\,a'\,o \perp^r a'\,b'$, and the horizontal trace by drawing $o\,q \perp^r a\,b$; o being the point where the traces intersect on the line of level $r\,s$. This plane $p\,o\,q$ will cut the sphere in a circle, which will contain the angles of the square, formed by the face of cube $\perp^r$ A B through A, the point A being one of them. Let this circle be turned about the horizontal line M N, M, N being points common to the plane $p\,o\,q$, and the sphere $m\,n\,s$ will be two points in the circle required; and A 'constructed' will be a third. Therefore a circle described through the three points, M, N, and A, will be the circle required. In this circle describe the square A E D C, the side of which will be $= a'\,b'$. Next, suppose the plane to resume its first position and determine the projection of the face $a\,e\,d\,c$ as in previous examples, and the cube may be completed without further difficulty.

52. To draw the plan of a cube when suspended from any point in one of its faces.

(Plate XV.) Let $a\,b\,c\,d$, $a'\,c'\,\gamma'\,a'$ (No. 1) be respectively the horizontal and vertical projections of the cube when one of its faces is horizontal. Let $(p'\,p')$ be the point of suspension in the top face. Let $(g\ g')$ be the centre of gravity of the cube. Join $g\,p$, $g'\,p'$. The vertical plane of projection must be taken parallel to the line G P, so that if another projection be taken on a ground line perpendicular to $g'\,p'$, it would be the plan required, because the line passing through P, the point of suspension, and the centre of gravity G, would be vertical.

Or we may suppose, as in No. 2, the whole solid to revolve about a horizontal line R S (taken at right angles to the vertical plane containing G P, and passing through any point O taken in the direction of G P), until the line G P is vertical—that is, until $g'\,p'$ is perpendicular to $x\,y$. It is evident that each point, as C will describe a vertical circle, having $c\,c$ for its horizontal projection, $\perp^r$ to $r\,s$ and $c\,c'$ described with centre o' for its vertical projection. The distance $g'\,c'$ is the same in No. 1 and No. 2, and similarly for the other points.

The plan of c is now found by drawing the perpendicular $c'c$ to xy to intersect $c'c$ in c.

53. The above problems have all had reference to a cube, in order to fix the principles of the subject more firmly in the student's mind; but the same problems are equally applicable to any other right-lined solid, and we should recommend the student before proceeding farther to well exercise himself by applying them to the other regular solids, and to given pyramids and prisms.

54. In a tetrahedral solid A B C D (Fig. 1, Plate XVI.) the angles between the edges A B, A D, A C are $\angle$ B A D = 43°, B A C = 54°, and D A C = 32°. The lengths of the edges A B, A D, and A C are respectively 3·05″, 1·8″, 3·7″.

Draw the plan of the solid when the face A B C is horizontal, and find the dihedral angles between the planes A B C, A D C, and A D B.

Draw the horizontal face A B C, making A B = 3·05″, A C = 3·7″, and the angle B A C = 54°. Next, suppose the two adjoining faces B A D, C A D to be turned about the edges A B, A C until they become horizontal. Thus, make angle B A D′ = 43°, and $\angle$ C A D″ = 32°, and make A D′ = A D″ = 1·8″.

Now in revolving the faces D′A B, D″A C about A B, A C respectively until A D′, A D″ unite, D′ will traverse the vertical circle projected horizontally in the straight line D′ E D drawn perpendicular to A B, and D″ the vertical circle D″ F D. Hence the point D, which is the point common to both circles, will be the vertex of the solid. Join D A, D B, D C, and the plan is completed.

To determine the dihedral angle between the faces A B C, A B D, suppose them cut by the vertical plane D E at right angles to their line of intersection A B, and therefore to each face. The distance from E to vertex D is evidently equal to E D′, therefore in turning the vertical plane E D about its horizontal trace E D, draw D d' $\perp^r$ to E D, and, with E as centre and radius equal to E D′, describe a circle cutting D d' in d'. Join d' E; then d'E D is the angle required.

To determine the angle between the faces A B C, A D C, it is only necessary to make D d'' $\perp^r$ to D F and = D d'. Join F d''; then $\angle$ d'' F D is the angle required.

Lastly, to determine the dihedral angle between the faces A D B, A D C Suppose a plane to pass through D at right angles to A D, their line of intersection.

Draw D′ G perpendicular to A D′ cutting A B in G. Then G will be the

point of intersection of the cutting plane with A B and therefore a point in its horizontal trace; and since the plane is perpendicular to A D, its horizontal trace must be perpendicular to the plane of A D. Draw G H perpendicular to A D cutting A C in H. Join G H, D H. Then the angle G D H is the angle required. The real size of this angle is determined by turning the triangle G D H about the horizontal line G H. The point D will traverse the vertical circle A D δ, and the real length of G D being G D′ with G as centre and G D′ as radius, describe a circle cutting D δ in δ. Join δ G, δ H, then the $\angle$ G δ H is the angle required.

55. In a tetrahedral solid A B C D (Fig. 2, Plate XVI.) A B = 3″, A D = 2″, A C = 2·25″, the angle B A C = 40°, $\angle$ D A C = 50°, and the dihedral angle between the faces A D C and A B C is 60°; determine the plan of the solid when resting on the face A B C, also determine the real angle between the edges AD, AB, and the dihedral angles between the faces A D B, A B C, and between A D B, A D C.

After having drawn the horizontal face A B C as given, draw A D″ = 2″ and making $\angle$ 50° with A C supposed turned down about A C into the horizontal plane. In this rotation D″ will describe the vertical circle D″ F D drawn perpendicular to A C. Again, the line in which the vertical plane D F cuts the face A D C will make with D F an angle equal to the inclination of A D C. Suppose this line turned down about D F, then D F d'' will be = 60°, and F d'' is evidently equal to F D″ or the distance of D from F; draw d'' D at right angles to D F; then D will be the plan of vertex of tetrahedron; join D A, D B, D C, and the plan of solid is complete. To determine the angle D A B, it is only necessary to suppose D turned about A B into the paper, describing the vertical circle D E D′, perpendicular to A B, making A D′ = A D″, and $\angle$ D A B will be the angle required.

Knowing the three angles the edges meeting in A make with each other, the remaining constructions will be precisely the same as those used in the previous problem, and as the same letters are used in both figures, the description will equally apply to both.

56. In a tetrahedral solid A B C D (Fig. 1, Plate XVII.) A B = 2·6″; A D = 1·74″; A C = 3·05″; $\angle$ B A C = 34°; $\angle$ C A D = 54°; and the dihedral angle between the faces A B C and A D B is 70°; determine the plan of the solid when resting on the face A B C; also determine the real angle between

the edges A D, A B, and the dihedral angles between the faces A B C, A D C and A D B, A D C.

After having drawn the horizontal face A B C as given, draw A D″ = 1·74″ making with A C the given angle 54°, supposed to be turned down into the horizontal plane. D describes the vertical circle K F D″ perpendicular to A C. Now since the face A D B is inclined to the face A B C at ∠ 70°, if F L is drawn perpendicular to A B and ∠ F L M be made = 70°, and M L produced to meet the perpendicular to F L at F in M, the triangle F L M will represent the section of the vertical plane F L with the plane A D B, and F M the height of F, if supposed to lie in the plane A D B. Now the vertex of the solid, D must lie somewhere in the plane A B D and the vertical plane K F D″; it will, therefore, lie in their common line of intersection. Make F N perpendicular to K F D″ and = F M the height of F as before found; and join K N, this is the line of intersection required when turned down about K F D″. The distance of D from A C, or the radius of the vertical circle it describes, is = F D″; therefore, with centre F and radius F D″, describe a circle to cut K N in d''; and, lastly, draw the projection d'' D perpendicular to K D″ and meeting it in D; join D A, D B, D C, and the plan of the solid is completed. The angle between the edges A B, A D, is found by constructing or turning the plane A D B about A B. The point D describes the vertical circle D E D′, drawn perpendicular to A B. Now, when the point D is constructed, its distances from the stationary points K and A will remain the same; these two distances are respectively A D″ and K D″; therefore, if with centre A and radius A D″ a circle is described, and with centre K and radius K d'', another circle is described, they will intersect in the required point D′, which will therefore lie on the line D E D′. Then the angle D A B will be the angle required. The construction required for finding the remaining angles is shown in the figure, and is precisely the same as that required in the last two examples.

57. An oblique parallelopiped A B C D E F G H (Fig. 2, Plate XVII.) has one face A B C D, a parallelogram of 2″ and 1·25″ sides and an ∠ 60°; another face, A D E F, has its sides of 2″ and ·75. The dihedral angle contained by the faces A B C D and A D E F is 70°, and that contained by A B C D and A B G F is 55°; draw the plan of the solid and determine the third dihedral angle made by the planes A D E F and A B G F. (King's College Examination paper).

After having drawn the parallelogram A B C D on which the solid is supposed to rest, draw the plane D K N inclined to the horizontal at an angle of 70°,

Fig. 1.

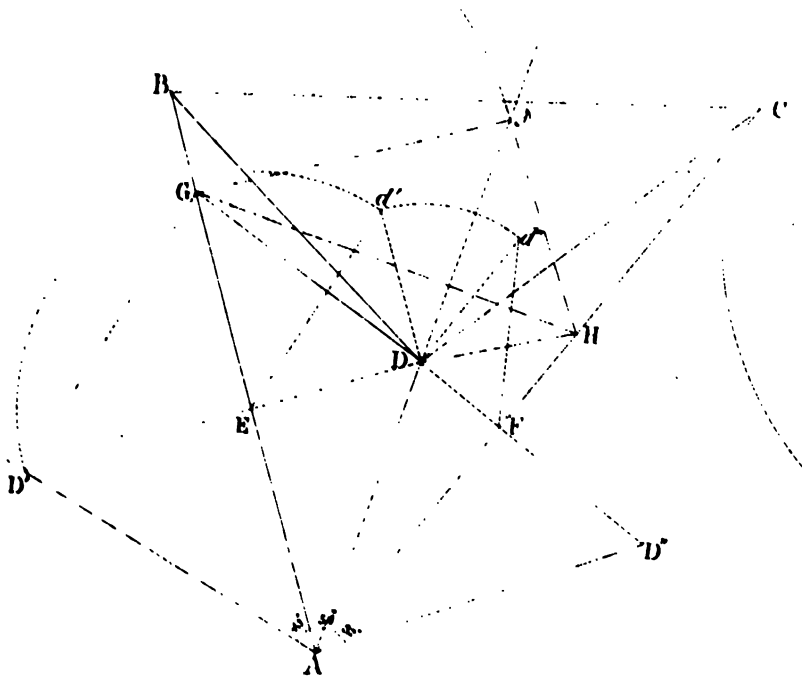


Fig. 2.

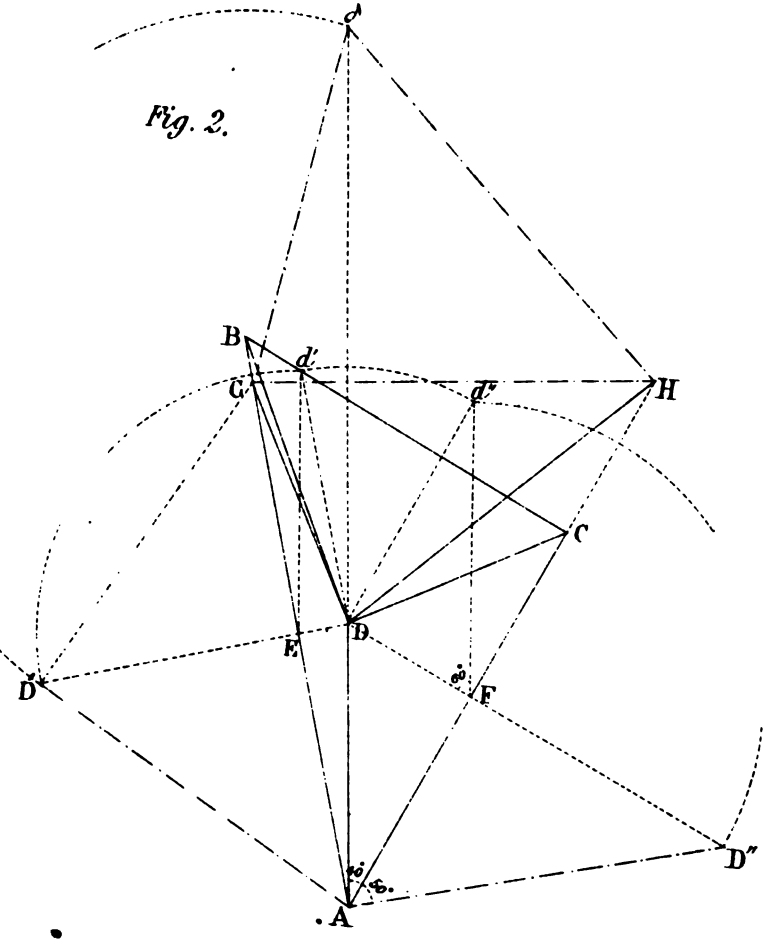


Fig. 1.

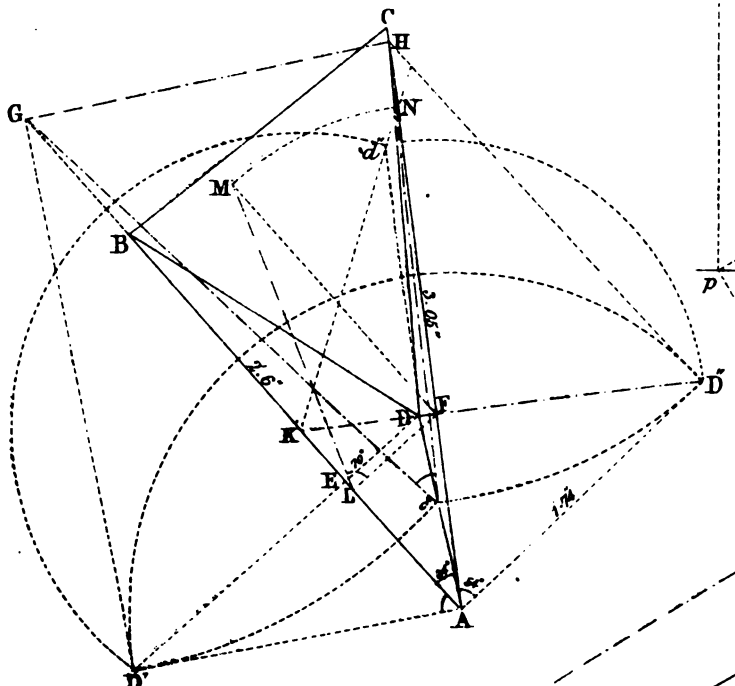
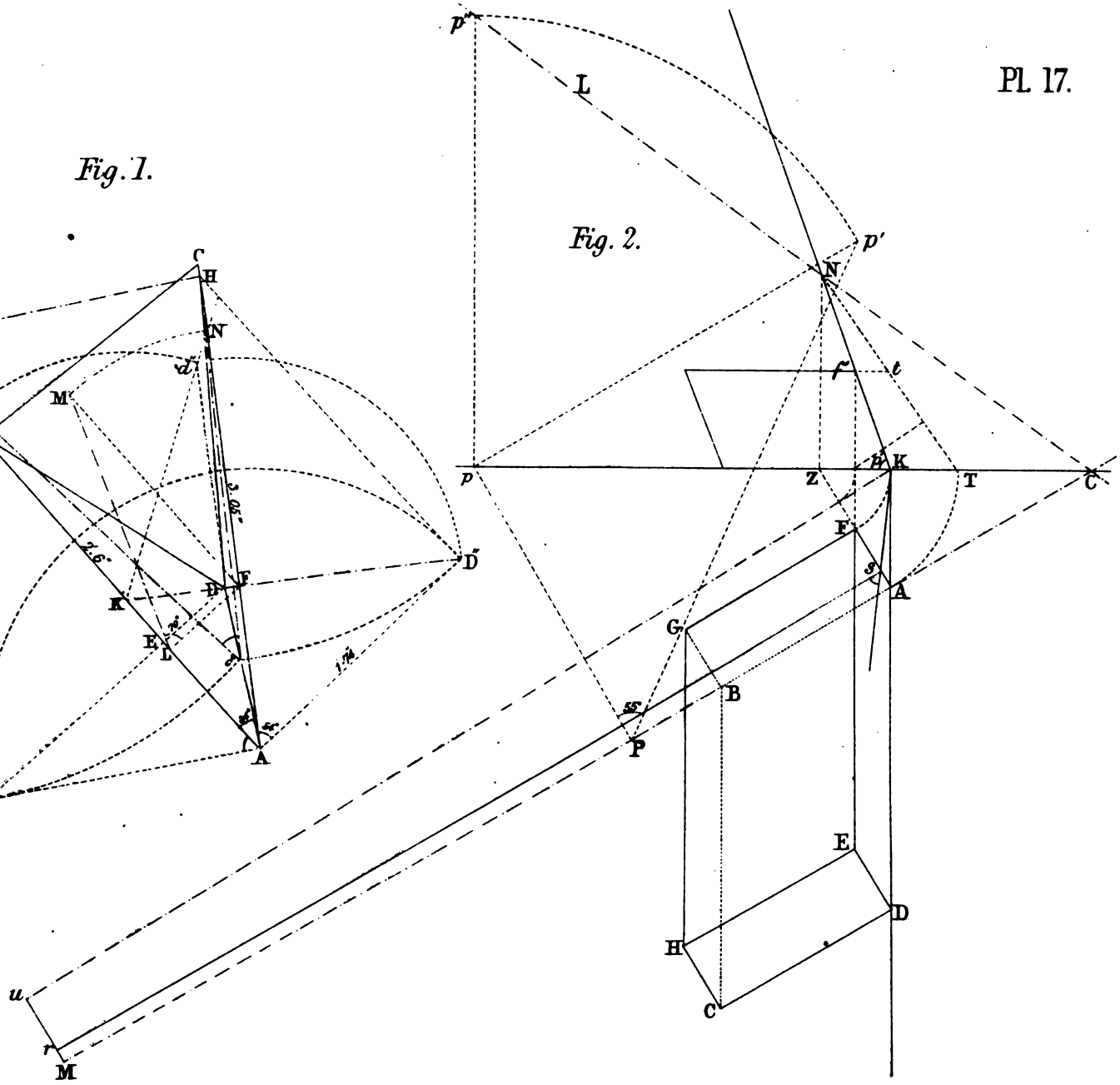


Fig. 2.



and perpendicular to the vertical plane of projection, so that the vertical trace K N is inclined to the ground line p K O at the $\angle$ 70°, and the horizontal trace K D is at right angles to it.

Next, to find the vertical trace of the plane of face A B G F which is to be inclined at 55°, and of which A B is the horizontal trace. Take any point P in the horizontal trace A B, and draw P p perpendicular to A B to meet the ground line in p; draw $p\,p'$ perpendicular to P p and P p' inclined to P p at an angle of 55°, to meet $p\,p'$ in p'. Then $p\,p'$ is the height of the point in the vertical p common to the sloping plane and the vertical plane of projection, and is therefore a point in the vertical trace required. Draw $p\,p''$ perpendicular to the ground line, and equal to $p\,p'$ and join p'' O; p'' O is the vertical trace required. Next find the line of intersection of the two planes L O M and N K D.

The plan of this line A Z will be the direction of the plan of the third edge of the solid. To find the length of the projected edge, suppose the line to revolve about the vertical at Z until it coincides with the vertical plane of projection. The point A will then be at T and the line Z A projected vertically in N T. Make T t equal to the given length, and then revolve the line back again into its former position, the point t will describe the horizontal circle represented in its vertical projection by the line $t\,f'$ and the point f', where it cuts the line N K will be the vertical projection of the end required, which can now be easily projected in the point F in the line A Z, and the plan of the solid completed by drawing B G, C H, D E, parallel and equal to A F.

The method by which the dihedral angle between the two faces A D E F and A B G F is indicated in the figure, and as it depends on the principles explained in Art. 54, will require no further explanation.

CHAPTER V.

ON SECTION PLANES AND THE INTERSECTIONS OF SOLIDS.

The intersection of a prism, cylinder, pyramid and cone by a given plane. The real form of section, and development of the solids.

58. To determine the projections of the section made by a given plane R S T (Plate XVIII.) with the twelve-sided prism a, b, c . . . m in plan.

The line joining two of the opposite corners of the base of prism $a\,g$ has been taken at right angles to the horizontal trace of plane S T. Of course the horizontal projection of the section required is the polygon $a\,b\,c$ . . . m itself, as the edges of the prism through these points are vertical, it therefore only remains to find the vertical projection of the points in which the plane R S T cuts the vertical edges aforesaid.

This may be done by either of the following methods. First, suppose a horizontal line to pass through a point of section, lying in the given plane; $k\,o\,d$ V drawn parallel to S T is the horizontal projection of an horizontal line through the points of section D and K which are on the same level, because they are the same distance from S T. The point in which the line K D meets the vertical plane of projection, must also be a point in the vertical trace R S of given plane, and is found by drawing V Y perpendicular to the ground line to meet R S in Y. Through Y draw Y $d'\,o'\,k'$ parallel to the ground line, it will be the vertical projection of the horizontal line D K and the points $d'\,k'$ in which it cuts the vertical projections of the vertical edges $\theta\,k'$, $\delta\,d'$ will be the vertical projections of the points D K required.

The other construction mentioned above is shown in the construction to find the points A, G. This is to suppose a vertical plane to contain the vertical edges through the opposite points A, G, and then find the vertical projection of its line of intersection with the given plane R S T. (24). This line R $a'\,o'\,g'$

must evidently contain the vertical projections of the points required, and since they must be also in the lines $a\,a'$, $\tau\,g'$, they will lie in the points of intersection a' and g'.

By pursuing either of these methods with respect to the remaining points the vertical projection of the polygon of section $a'\,b'\,c'\,\ldots\,m'$ is determined.

Next to determine the real form of the section. This may be done by 'constructing' the plane containing it, or by turning it about its horizontal trace until it coincides with the horizontal plane of projection, or about its vertical trace until it coincides with the vertical plane of projection. The latter plan has been adopted in this instance as being the more convenient.

Thus to find the position of the point M, it will be observed, this point describes a circle perpendicular to the plane of projection, and will therefore be projected in the straight line, $m'\,u$ M perpendicular to the hinge-line R S, and the distance u M will be equal to the radius of said circle or the distance of M from R S. This distance is found by setting off $\mu\,m$ the distance of M from the vertical plane of projection from u to r along R S and then the distance $v\,m'$ or the hypothenuse of the right-angled triangle $v\,u\,m'$ will be equal to the radius required, and this set out from u to M will be the position of the point of section M when turned into the vertical plane of projection. And by applying this method to all the other points the real form of the section A B C . . . M is determined.

The polygon $a'\,\beta'\,\gamma'\,\ldots\,\mu'$ is the projection of the base $a\,\beta\,\gamma\,\ldots\,m$ when the solid has its plane of section in the vertical plane of projection as in A B C . . . M.

To find the projection of the point (o, ω) the centre of the base, it must lie on the line $\omega\,z'\,\omega'$ drawn perpendicular to R S; find the point $(z\,z')$ in which the perpendicular through (o, ω) cuts the given plane (28 and 29); and then deal with the point $z\,z'$ in the same manner as with one of the points of section as M, and thus the position of the centre ω' is determined. Join O, the centre of the polygon A B C . . . M with ω' and through A B C . . . M draw lines A a', B β', &c., parallel to O ω' to meet the projectors $a\,a'$, $\beta\,\beta'$, &c., drawn through a, β, &c., perpendicular to R S in a', β', &c., and so the base $a'\,\beta'\,\gamma'\,\ldots\,m'$ is found.

The real form of section may also be determined by the following method:—

Find the horizontal distance of any one of the points of section, as $k\,k'$, from the vertical trace of the section plane; this will be the horizontal line K' Y, K V, lying in the plane of section. The real length of this line is K V;

and as the point Y where it meets the vertical trace is fixed, if Y be taken as centre and K V as radius, the point K must lie somewhere on this circle, it must be also in the vertical circle projected in the line k' K, drawn perpendicular to R S, therefore it will lie in the point of intersection.

Plate XIX. shows the development of the above prism, when each face is turned about its line of intersection into the same plane with all the other faces, so as to form one plane surface, showing the real form of each face. The figure is easily deduced from Plate XVIII. by making the line $g\,a\,g$ equal to the sum of the twelve sides in the polygonal base of plan in Plate XVIII., dividing it into twelve equal parts in a, b, c, &c., and making the perpendiculars a A, b B, &c., equal to the heights of the corresponding vertical edges of the prism in Plate XVIII., and lastly, the lines joining the points A B C, &c., will give the development of the line of section.

59. To determine the projections of the section made by a given plane R S T with a given cylinder.

The several points in this section might be determined by either of the methods given in the preceding article by considering the cylinder as a polygonal prism with an infinite number of sides; but since we know that the section of a cylinder by a plane is an ellipse, it is better to find the major and minor axes of the ellipse, which is the projection of said ellipse, when the curve can be drawn in the usual manner.

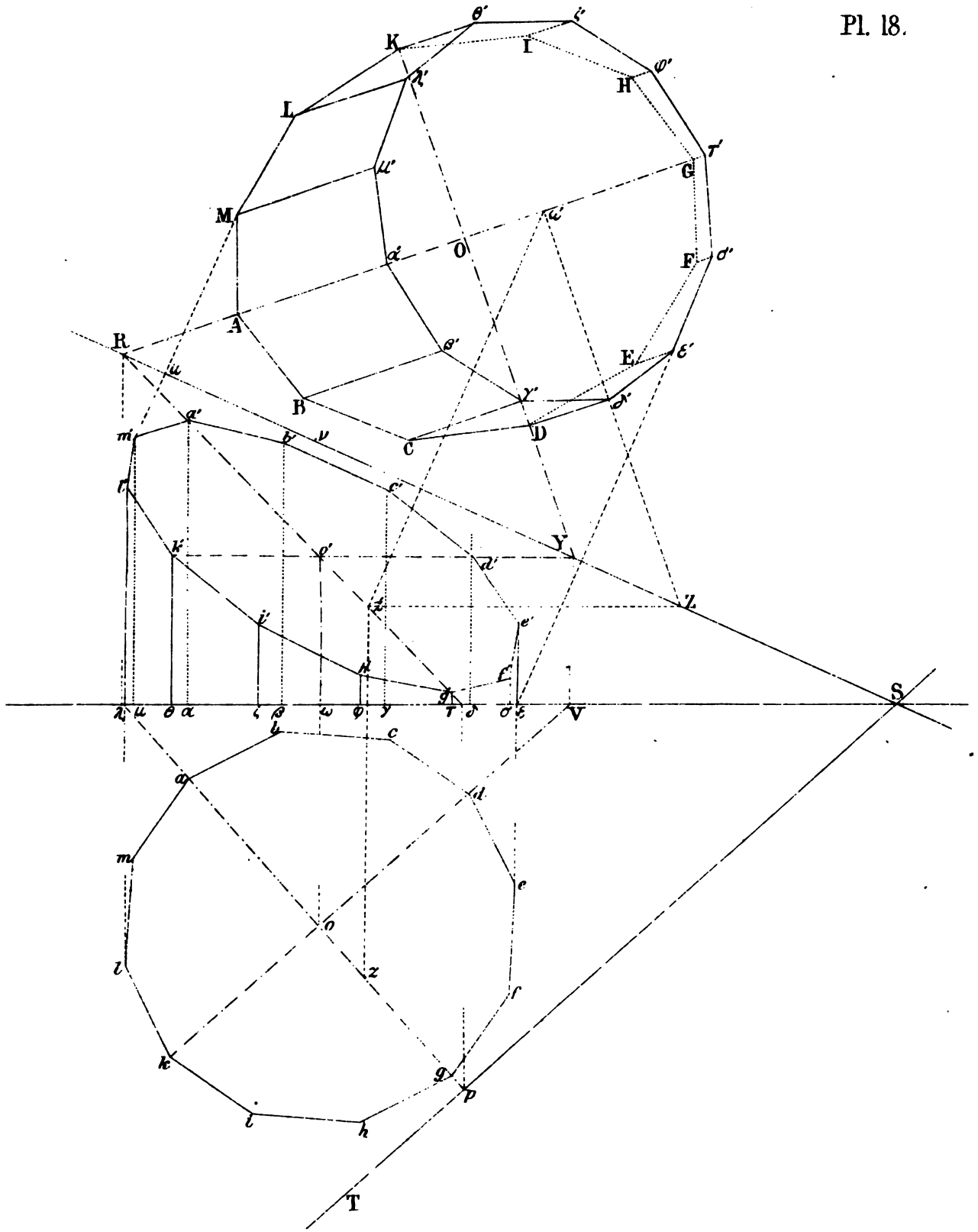
Of course, the plan of the section required, is the circle $a\,c\,b\,d$, and therefore the lines $a\,b$ and $c\,d$, drawn at right angles through the centre o, will represent the plans of two conjugate diameters of the elliptic section, and their vertical projections (determined as lying in the section plane R S T) $a'\,b'$, $c'd'$ will be conjugate diameters of the ellipse, which is the vertical projection of the section. If $c\,d$ be taken as in the plate, parallel to the horizontal trace T S, $c'\,d'$ will be a line parallel to the ground line, and distant from it equal to $x\,\delta$.

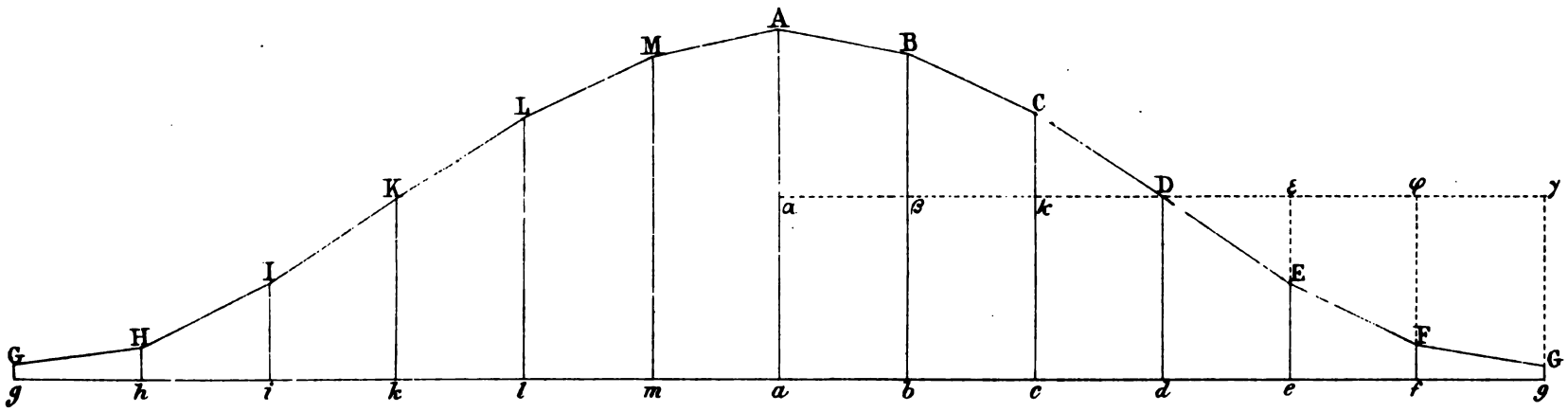
Having found the two semi-conjugate diameters, $o'\,a'$ and $o'\,c'$, the major and minor axes may be determined by the following construction, which is not given in the plate for fear of confusing the figure.

Produce $o'\,c'$ to l, making $c'\,l\,a$ third proportional to $o'\,c'$, $o'\,a'$.

Bisect $o'\,l'$ in x and draw $x\,z$ at right angles to $o'\,l$ to meet the tangent at c' (which is, by hypothesis parallel to $a'b'$) in z; with z as centre, and $z\,l$ or $z\,o'$ as radius, describe a circle, cutting the tangent at c' in t and s; join $o'\,t$ and $o'\,s$, these lines will represent the directions of the major and minor axis respec-

Pl. 18.


θ' ι' K I φ' H L λ' τ' G M μ' ω' O α' F σ' A R ω ε' E B ρ' γ' δ' a' m b' ν C D c' l' Y k' φ' d' Z i' e' h' f' S λ μ θ α ζ β ω φ γ τ δ σ ε V b c a d m e o l z r k g p h i T


A
M
B
L
C
K
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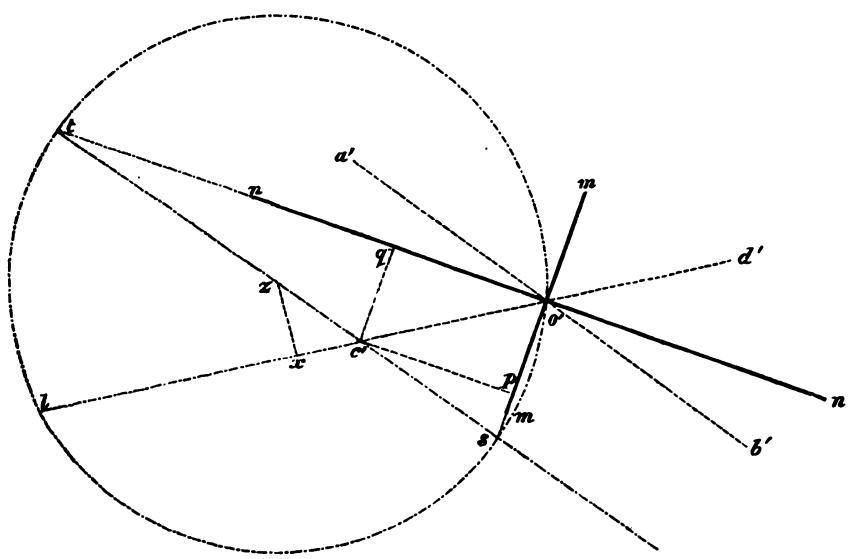
tively; to find their magnitudes, draw $c'q$ perpendicular to to', and $c'p$ perpendicular to so', and make $o'n$ a mean proportional to $o'q$ and qt, and $o'm$ a mean proportional to $o'p$ and ps, then $o'n$ and $o'm$ will be the respective lengths of the semi-major and semi-minor axes.

Having thus found the principal axes, the ellipse may now be completed by means of a slip of paper in the usual manner.

It is well to determine the productions of the points K, L where the ellipse touches the contour lines kk' and $\lambda l'$. These points are readily determined by drawing kl parallel to the ground line to meet the horizontal trace in y, and then determining the vertical projection of this line as lying in the section-plane R S T.

The real form of the section may be determined in a precisely similar man-



ner to that described for the prism, but as the figure is in this case an ellipse, it is best to obtain the major and minor axis.

Now, the major axis will, of course, be the longest line that can be drawn through the centre of the ellipse in the section plane. This line is obviously A B, which contains the highest and lowest points. The vertical trace of A B is e', and the horizontal trace f. Find the real length of E F by making eg equal to ef; then $e'g$ is the real length. With e' as centre. and $e'g$ as radius,

describe the arc $g\,h$, and with S as centre and Sf as radius, describe an arc, cutting the former in h, join $e'\,h$, this will be the direction of the major axis, which can now be found by drawing $a'\,a'$, $b'\,\beta'$ perpendicular to R s, meeting $e'\,h$ in a' and β'. Bisect $a'\,\beta'$ and ω', and draw $\gamma'\,\delta'$ at right angles to $a'\,\beta'$, making $\omega'\,\gamma'$, $\omega'\,\delta'$, each equal to the radius of the circular section of the cylinder.

Plate XXI. shows the development of the above cylinder. The figure is easily deduced (in the same manner as that described for the prism in Plate XIX.), from the preceding plate by making the line $b\,a\,b$ equal to the circumference of the circular base of the cylinder, dividing it into any number of equal parts, 1, 2, 3, 4, 5, &c., and the circular base of cylinder in the plan of Plate XX. into the same number of equal parts. The vertical lines, 1–1, 2–2, 3–3, &c., are made equal to the heights of the corresponding vertical lines on the cylinder. The curve drawn through the extremities of these lines will be the development of the curve of section.

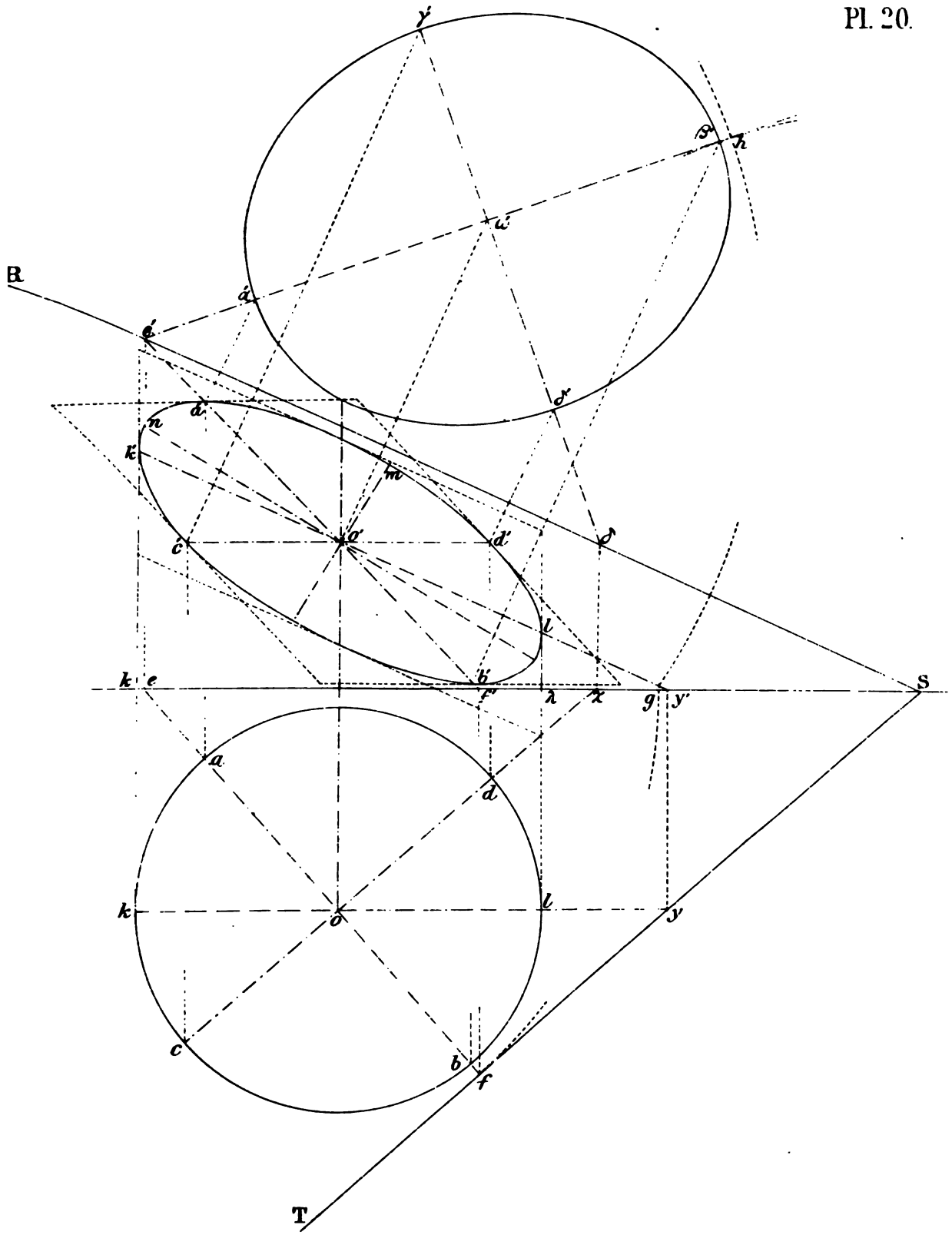
60. To determine the projections of the section made by a given plane R S T with a given pyramid.

(Plate XXII.) The pyramid in the figure stands on a regular twelve-sided polygonal base, but the principle of construction would equally apply to any other shaped pyramid.

There are two methods by which the projections of the required section may be found. The first is by means of vertical sections. Thus, let the vertical plane $z\,y\,y'$ contain the edges of pyramid A V and G V. The vertical projection $y'\,z'$ of the line of intersection of this plane with the given plane R S T, will contain the vertical projections of the points in which the given plane R S T cuts the edges of pyramid A V and G V; but these points must also lie on the lines $a'\,v'$ and $g'\,v'$; hence a' and θ' are the projections required, and from these points the horizontal projections a and θ may be deduced.

The other method is to suppose any horizontal plane $p\,n$ to cut the pyramid; it is convenient to take it half-way up the axis, because the horizontal projection of the section 1, 2, 3, ... 12, will bisect the lines $v\,a$, $v\,b$, $v\,c$, &c. Draw the horizontal projection $q\,t$ of the line Q T, in which the horizontal plane $p\,n$ cuts the given plane R S T.

Now, let us take any face of the pyramid D E V. Produce 5-4 to meet $q\,t$ in ρ, then ρ is the projection of a point common to the face D E V, and the given plane R S T, and is therefore the projection of a point in the line of intersection of these planes; similarly, by producing $d\,e$ to meet S T in r, we obtain the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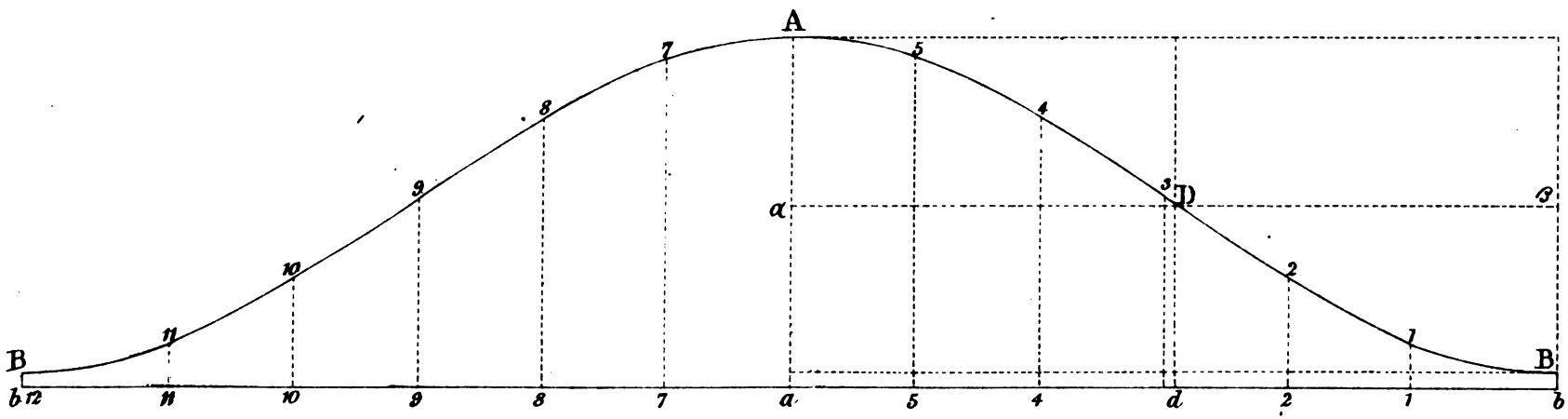
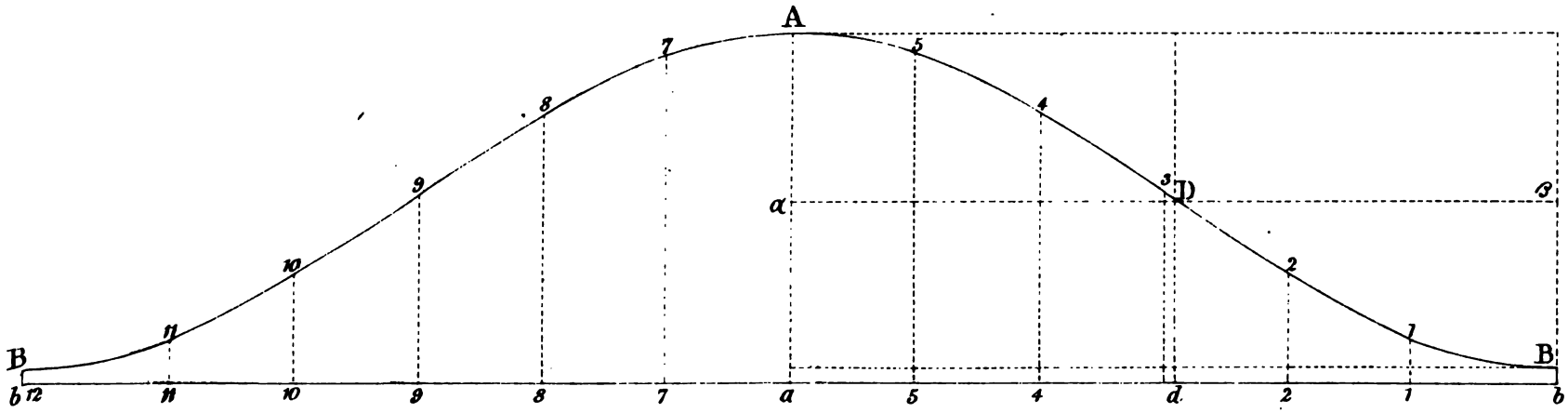
Pl. 21.

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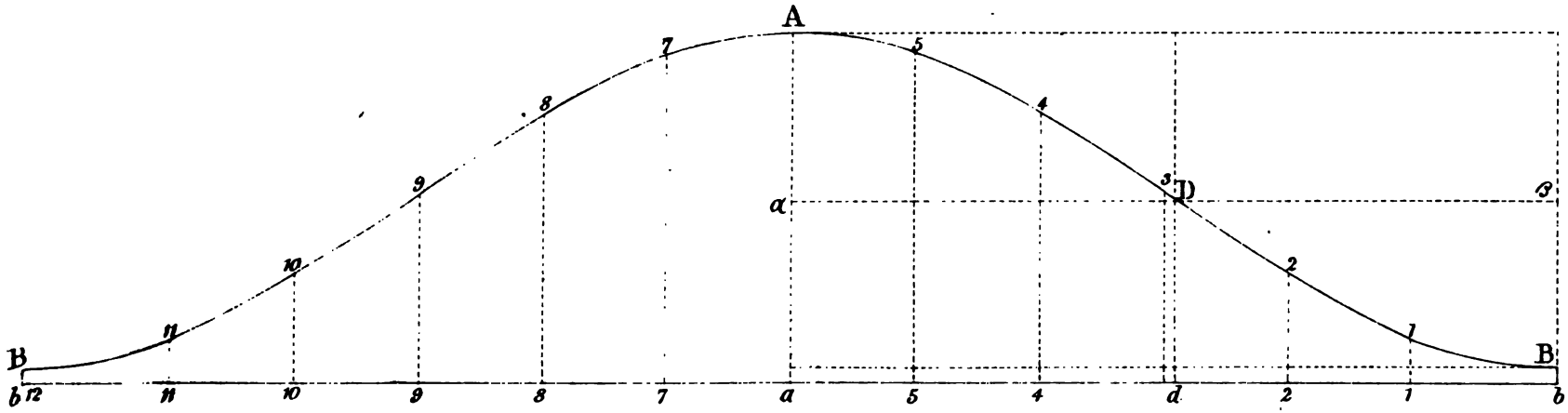
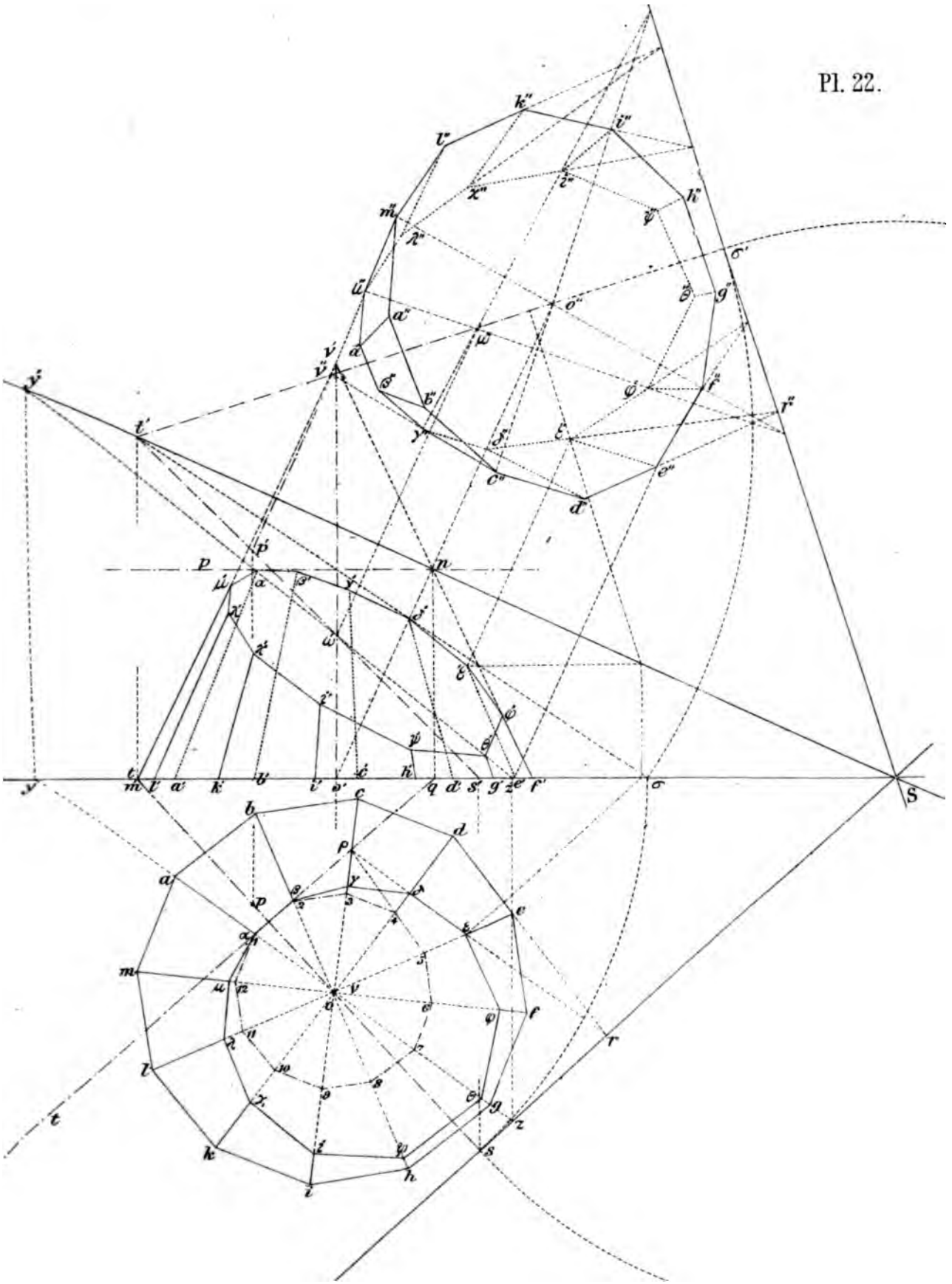
Pl. 21.

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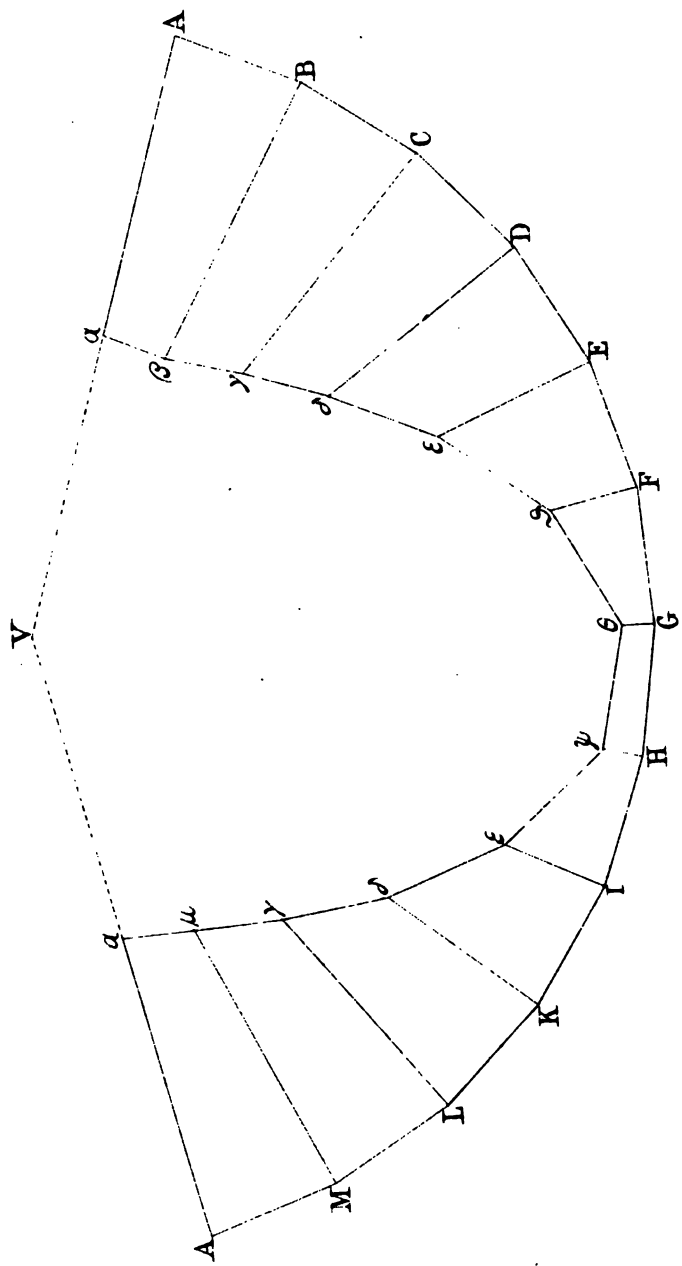
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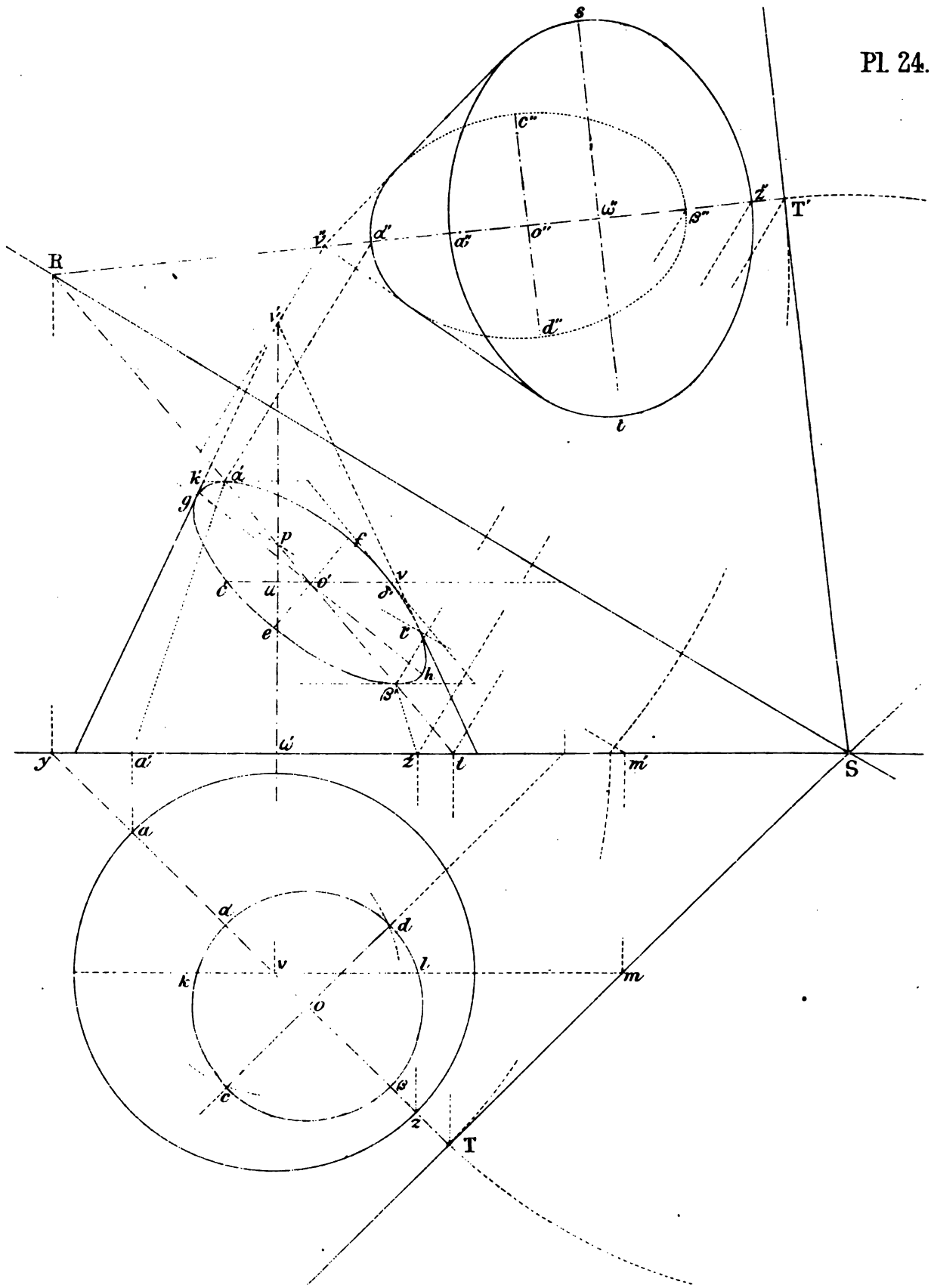
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projection of another point in the line of intersection required. Therefore, by joining ρr, we obtain the projection of the line of section $\delta \varepsilon$ required, and in like manner for the remaining faces. The upper part of the figure shows the method of obtaining the real form of section by revolving the section plane about its vertical trace until it coincides with the vertical plane of projection; but as this involves no new principle of construction, we leave it as an exercise for the student.

Plate XXIII. shows the development of the above pyramid as truncated by the section plane. With V as centre and radius equal to the real length of one of the edges, describe the circle A G A, and on it step out the equal lengths A B, B C, and C D, &c. Join these points with V and make A a, B β, &c., equal to the real lengths of the same portions as found by the two projections in the preceding plate. Lastly, join a, β, γ, &c., and the development is completed.

61. To determine the projections of a section made by a given plane R S T, with a given cone.

The conic section may be a circle, ellipse, parabola, or hyperbola, according as the cutting is respectively parallel to the base, inclined to the axis and all generating lines of the cone, parallel to a generating line of the cone, or parallel to the axis.

We will consider the general case, viz.: where the section plane is inclined to the axis and all generating lines. The section in this case is an ellipse, and the two projections will therefore be ellipses, of which it will be necessary to find the major and minor axes in order to construct them. To obtain the horizontal projection, find the line of intersection of the vertical plane T y R, drawn through the axis of cone perpendicular to the given plane R S T, with said plane; this line of intersection will contain the highest and lowest points (a, a') and (β, β') in the curve of section; hence, $a\,\beta$ will be the major axis of the ellipse in the horizontal projection; bisect $(a\,\beta,\ a'\,\beta',)$ in (o, o') and through O draw the horizontal line C D and find the points C, D, in which it cuts the surface of the cone. These are found by supposing a horizontal plane to contain the line C D, and drawing the plan of the circle formed by its section with cone; the points c, d, in which this circle cuts the line $c\,o\,d$, will give the minor axis of the ellipse, which may now be completed in the usual manner.

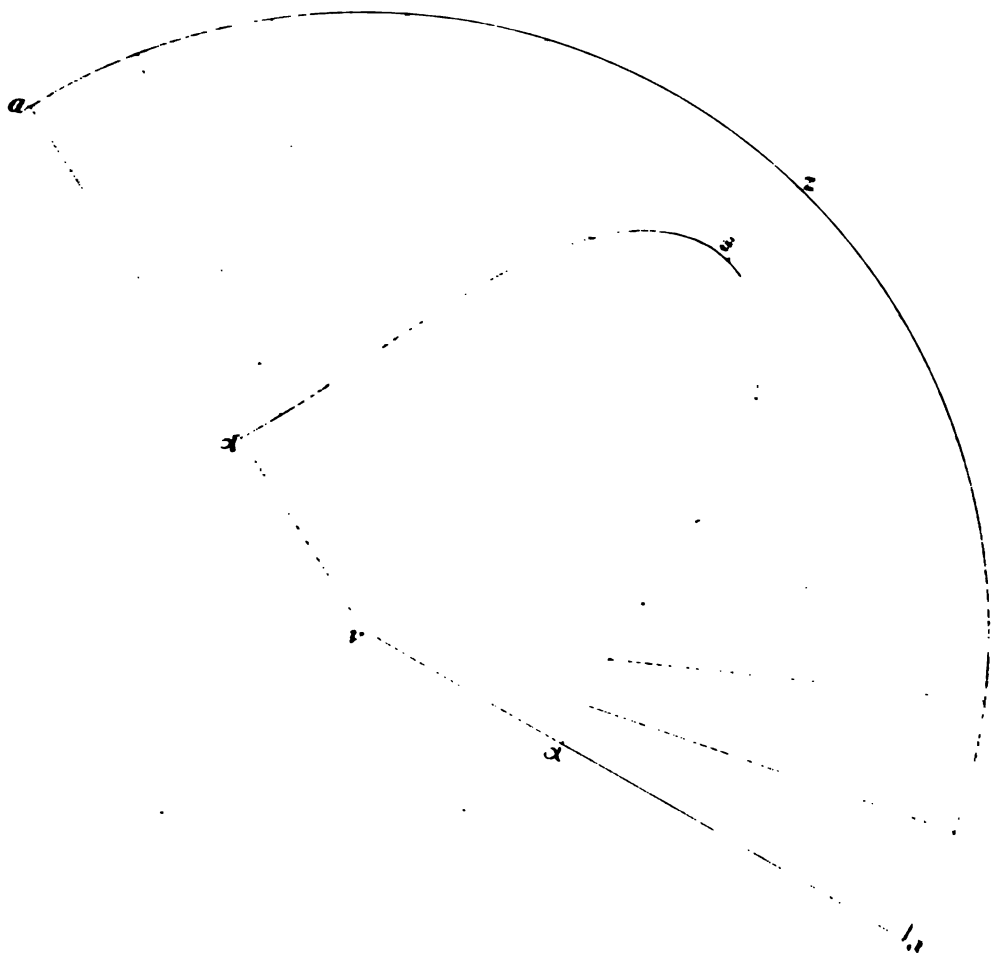
In the vertical projection the two diameters $a'\,o'\,\beta'$ and $c'\,o'\,d'$ will be two conjugate diameters from which the major and minor axes may be deduced by the method described in Art. 59.

In finding the real form of the section, it is well to find the position of the line of intersection of the vertical plane through axis and perpendicular to the plane R S T, with said plane, when said plane has been revolved about its vertical trace until it coincides with the vertical plane of projection; because, in this case, the line when constructed will contain the major axis of the ellipse, giving the real form of section, and the minor axis of the base of cone when resting in its new position. The minor axis of the ellipse of section is found by making $o'' c''$ equal to $o\,c$ or $o\,d$, and the minor axis of the ellipse of base by making $\omega'' s$ or $\omega'' t$ equal to the radius of circular base.

Plate XXV. shows the method of obtaining the development of the cone and its section, as given in the preceding plate. With centre v and radius equal to the length of a generating line of the cone, describe a circle $a\,z\,a$, and make its length equal to the circumference of the circular base of cone. Join $a\,v$, $a\,v$, then the figure $v\,a\,z\,a$ will be the development of the cone. Next, to draw on this development the line of section made by the given plane in the preceding plate; divide the circular base of cone into a number of equal parts, and its development $a\,1\,a$ into the same number, and join these points of division in each instance with the vertex v; then, by setting out on each of these lines their real length from the base to the point where they cut the required section (as $z\,\beta$ in Plate XXV., which is the real length of $z\,\beta$, $z'\beta'$ in Plate XXIV.), the development $a\,\beta\,a$ of the line of section can be readily determined.

62. To determine the section of a sphere by a given plane.

In problems connected with the sphere, it is often convenient to suppose the planes of projection to pass through the centre of the sphere itself, in which case the same great circle will represent both the horizontal and vertical projections of the sphere. This has been supposed the case in Plate XXVI., where z is the centre of given sphere taken on the ground line X Y, and $a'\,q\,c$ the given plane. On this supposition it is obvious that the points a', b', in which the vertical trace $a'\,q\,b'$ cuts the vertical great circle of the sphere, will be two points in the vertical projection of the section required, and their plans can be found as lying in the ground line X Y. Similarly, the points c, d, in which the horizontal trace $c\,q\,d$ cuts the horizontal great circle of the sphere, will be two points in the horizontal projection of the section required, and their vertical projections can be found as lying in the ground line X Y. Thus four points in the required section A B C D are at once determined. But as the plane section of a sphere must be a circle, and the projection of a circle is an ellipse, it is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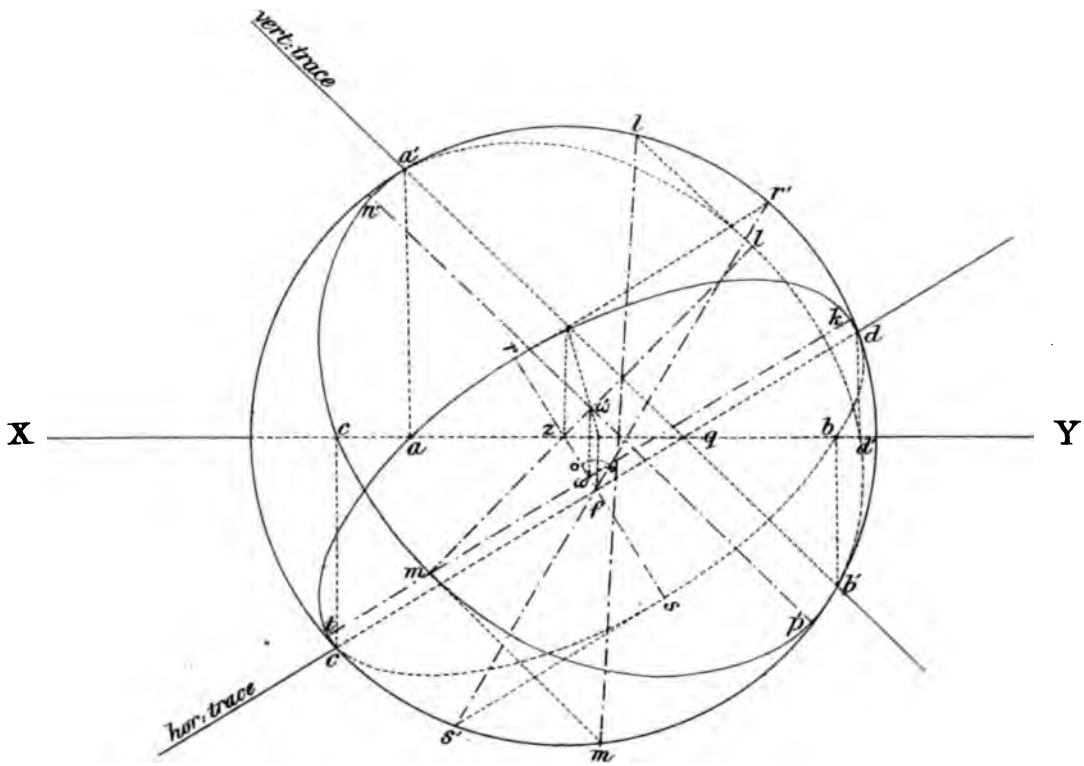
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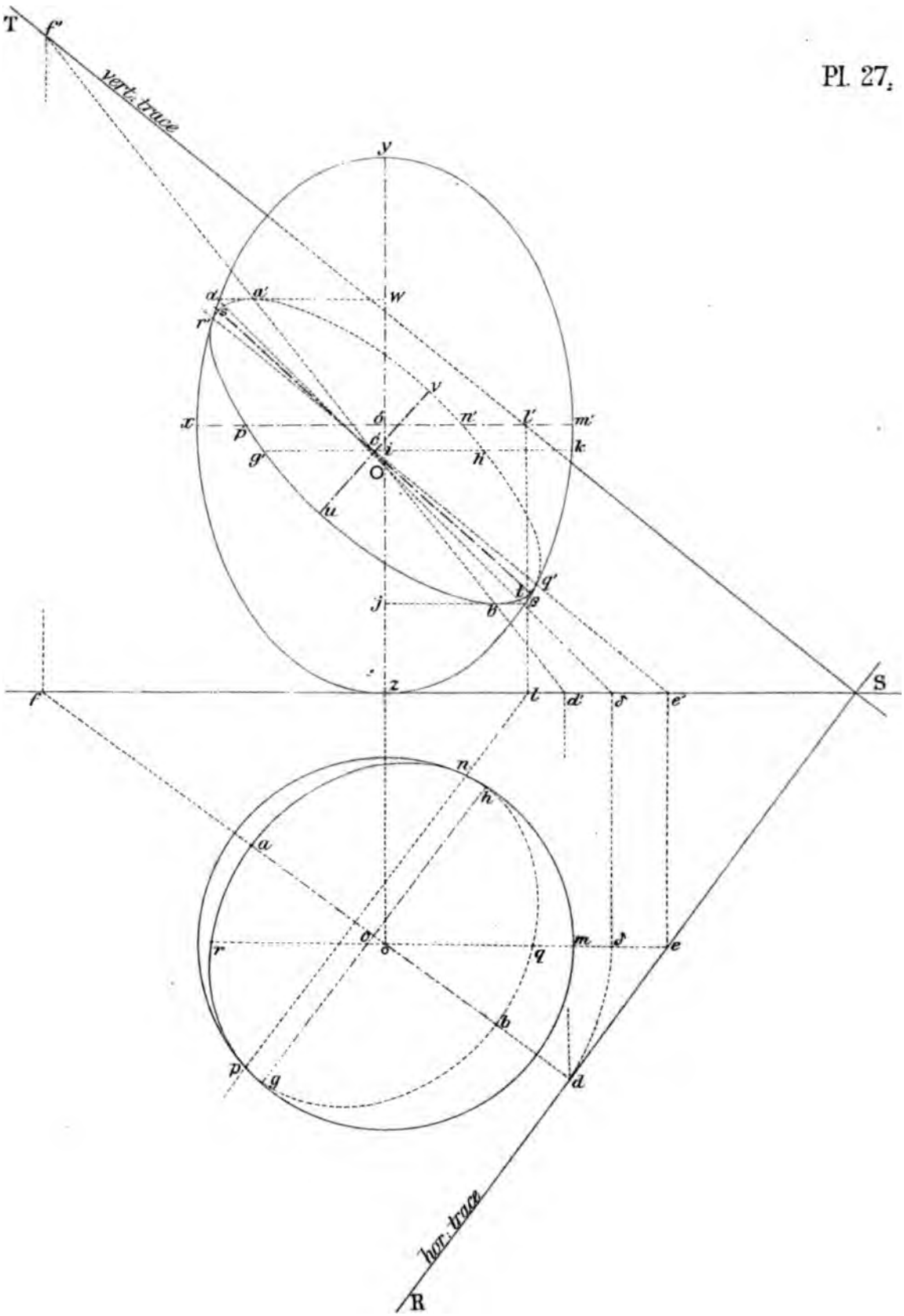
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necessary to determine the major and minor axis of the ellipse in each projection. Through z, the centre of the sphere, drop a perpendicular on to the given plane, intersecting it in the point ω, ω' (28 and 29). This point will be the centre of the required section. Take the line $r\,z\,\omega\,s$, drawn at right angles to the horizontal trace, as a line of level the same as X Y, or passing through the centre of sphere, and determine the profile r', s' of the given plane. This may readily be done, as we know the height $a\,a'$ of a point A in the given plane above the line of level. The points $r'\,s'$, in which this line cuts the great circle of sphere, will give the extreme points of the elevation of the section when $r\,s$ is taken as a line of level; and the places of these points r and s will give the minor axis of the ellipse, or horizontal projection of the section. The major will be a horizontal line passing through its centre ω, and equal to $r'\,s'$, which is the real length of the diameter of the circular section. The major and minor axes of the vertical projection are determined by a precisely similar construction, and are sufficiently indicated in the figure.

63. To determine the section of an ellipsoid by a given plane.

Plate XXVII. Let $z\,y, o$ be the projections of vertical axis of ellipsoid; R S T the given plane. First, determine the line of intersection F D of given plane R S T with the vertical plane $f'f\,d$, passing through the axis of ellipsoid, and perpendicular to the given plane. This line of intersection F D will contain the highest and lowest points of the required section, and also the point o, O, where the plane cuts the axis. To obtain the projections of the points where the line F D meets the solid, suppose it to revolve about the vertical axis of ellipsoid, always retaining the same inclination, until it becomes parallel to the vertical plane of projection; the point d describes the horizontal circle $d\,\delta$, and, since the point O in the vertical projection will not move, the line $a\,O\,\delta'$ will be the vertical projection of the line of intersection in its new position, and the points α and β, where it meets the vertical contour of the ellipsoid, will give the horizontal distances $w\,a, j\,\beta$ of the points of section required, from the axis, and their places may now be found by making $o\,a$, $o\,b$ respectively equal to these distances; and the points a' and b', in which $f'd'$ intersects $a\,w, j\,\beta$, will give the vertical projections of said points.

Bisect A B in the point C, and through C draw the horizontal line G C H, and find where this line cuts the solid. The places of these points are found by making $c\,g$ and $c\,h$ each equal to the radius $i\,k$ of the horizontal circle which is the section of the ellipsoid made by the horizontal plane containing G H. Then

$a\,b$ and $g\,h$ will be the minor and major axes of the horizontal projection of the required section. This ellipse may now be constructed. It is useful to determine the exact points n, p, in which it touches the horizontal contour of the ellipsoid. These points are given by determining the line of intersection of the plane of the greatest horizontal circle of ellipsoid with the given plane. The points N, P, in which this line meets the horizontal circle of ellipsoid through centre of axis, will be those required.

As we have the two conjugate diameters $a'\,b'$ and $g'\,h'$ of the vertical projection of the section known, the major and minor axes can be deduced by the method shown in Art. 59. The points r', q' where this projection touches the vertical contour are determined by finding the line of intersection of a plane containing axis, and parallel to the vertical plane of projection, with the given plane R S T, as is shown by the lines $r\,o\,q\,e$, $r'\,o'\,q'\,e'$, the projection of the line required.

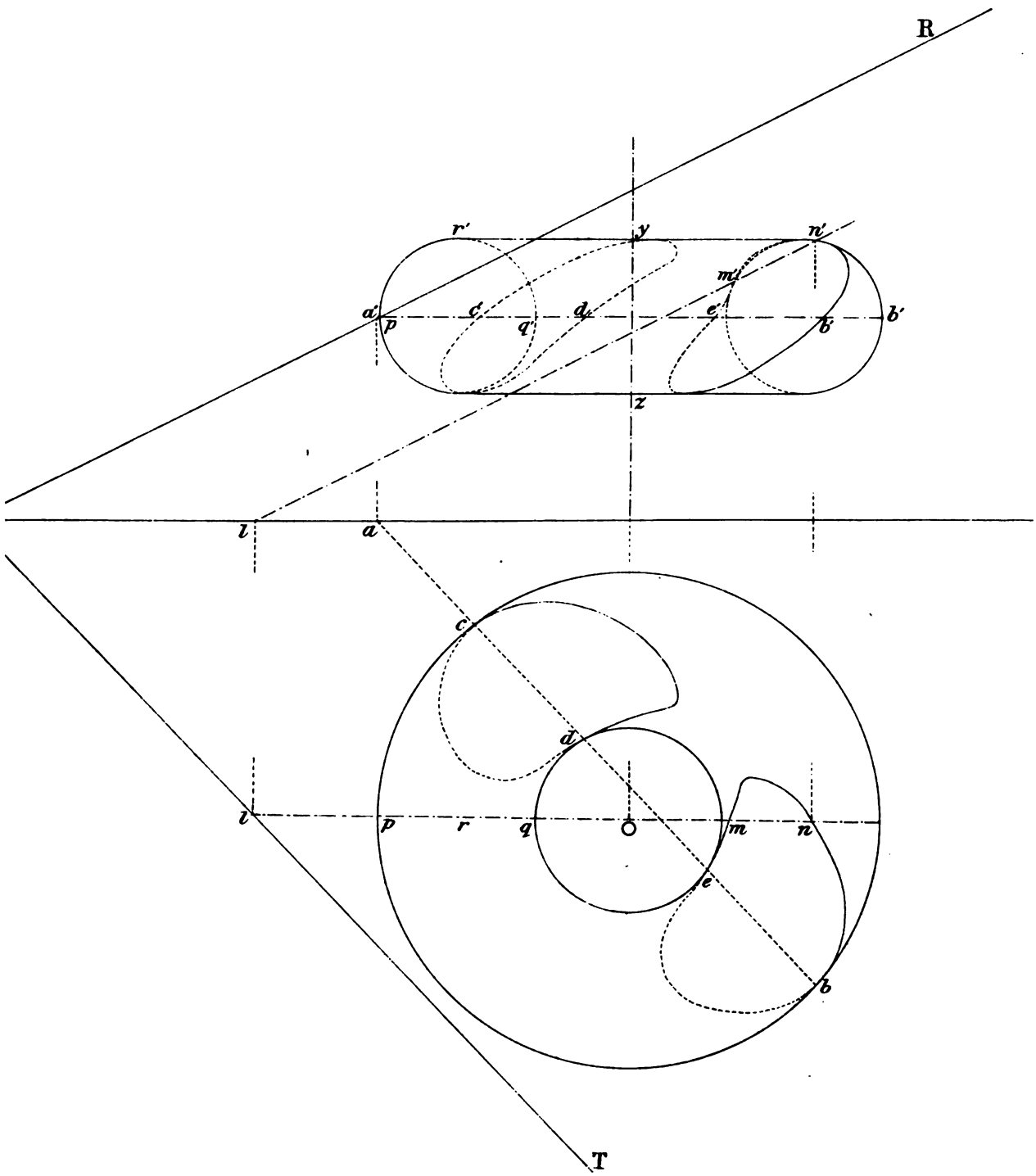
64. To determine the intersection of a given plane and a circular annulus. (Plate XXVIII.)

An annulus is formed by the revolution of the vertical circle P Q about the verticle straight line $(o, z\,y)$; the circle and straight line always being in the same plane.

Plates XXVIII., XXIX., and XXX. show three different positions of the cutting plane R S T; the construction, however, in each case is similar. Suppose a series of auxiliary horizontal planes cutting both the plane and annulus. Each of these horizontal planes will cut the annulus in two circles (excepting the case in which the horizontal plane touches the annulus), and the plane in a horizontal straight line, and therefore parallel to the horizontal trace of the given plane S T. The four points common to the two circles and straight line will be four points in the intersection required. Thus, in the figure we have shown only the horizontal circle cutting the centre of the axis $z\,y$, the plan of the cutting circles representing the contour of the horizontal projection. The four points, c, d, e, b, where the line of intersection of these two planes cuts the circles, will be four points in the intersection required.

In Plate XXIX. the cutting plane R S T is also a tangent plane to the annulus at the point F, on the minor or convex surface.

Plate XXX. represents the intersection made by the same plane and annulus, when the plane makes a smaller angle to the horizontal plane of projection by revolving it slightly about the horizontal trace S T. The constructions re-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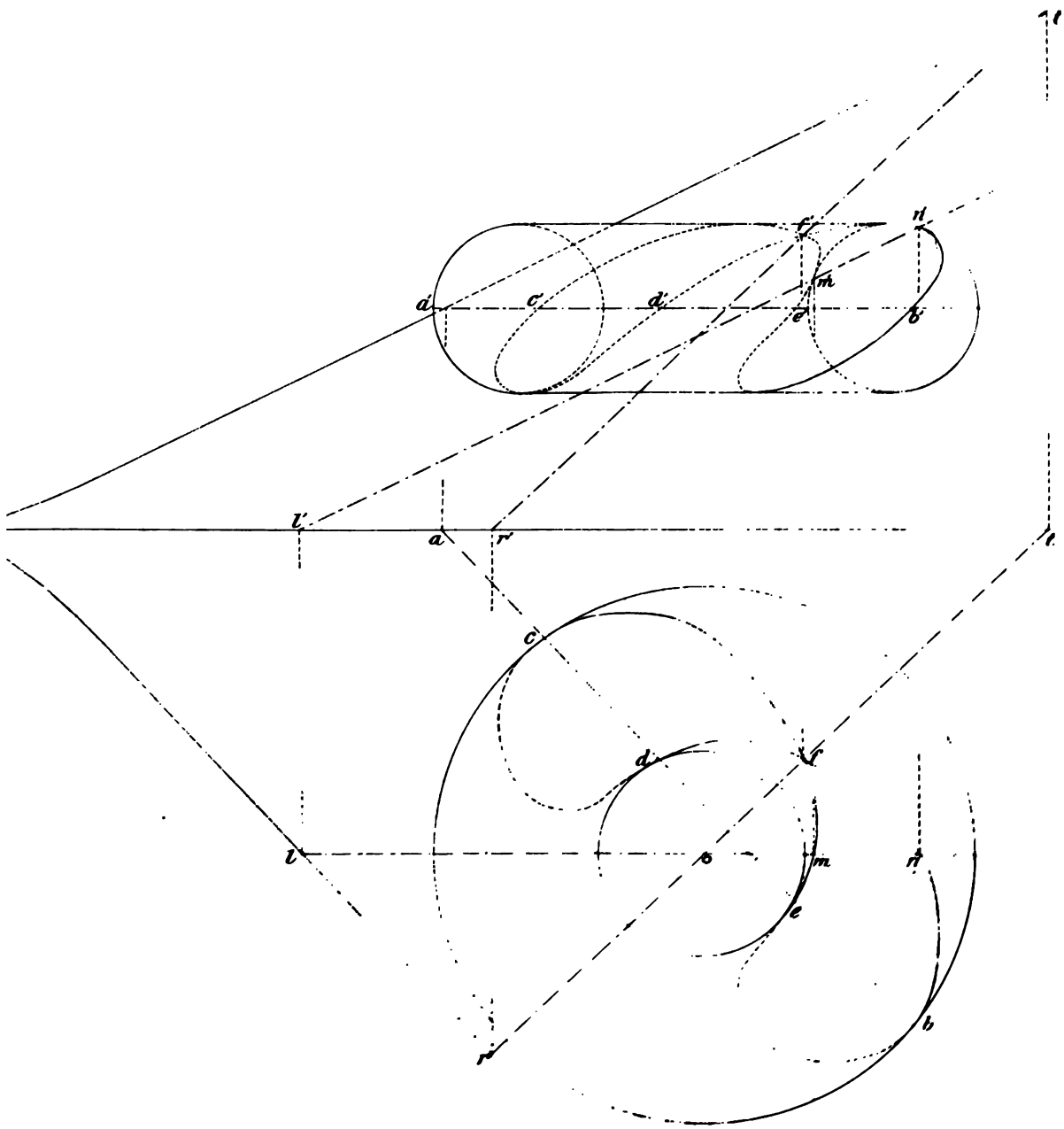

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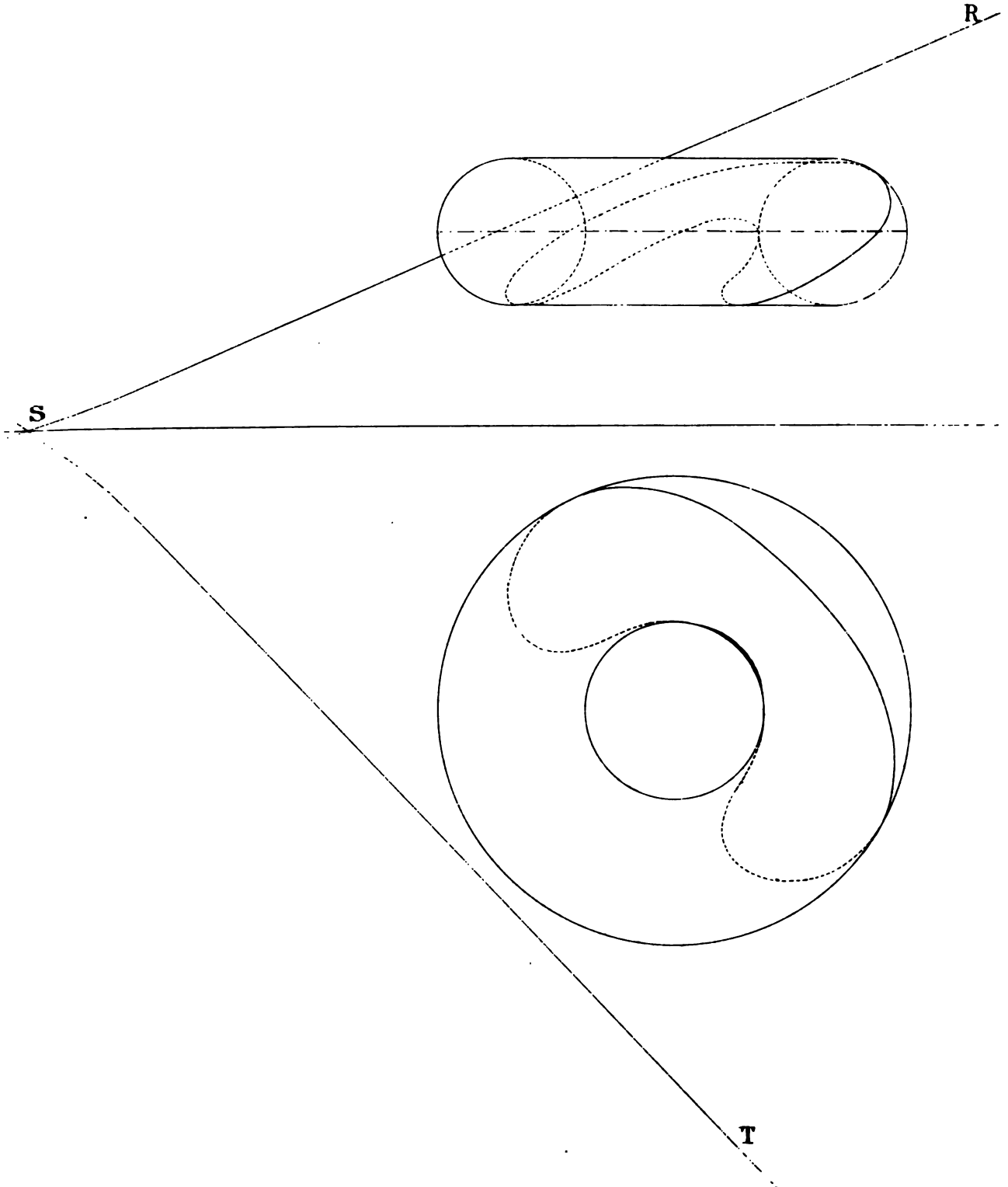
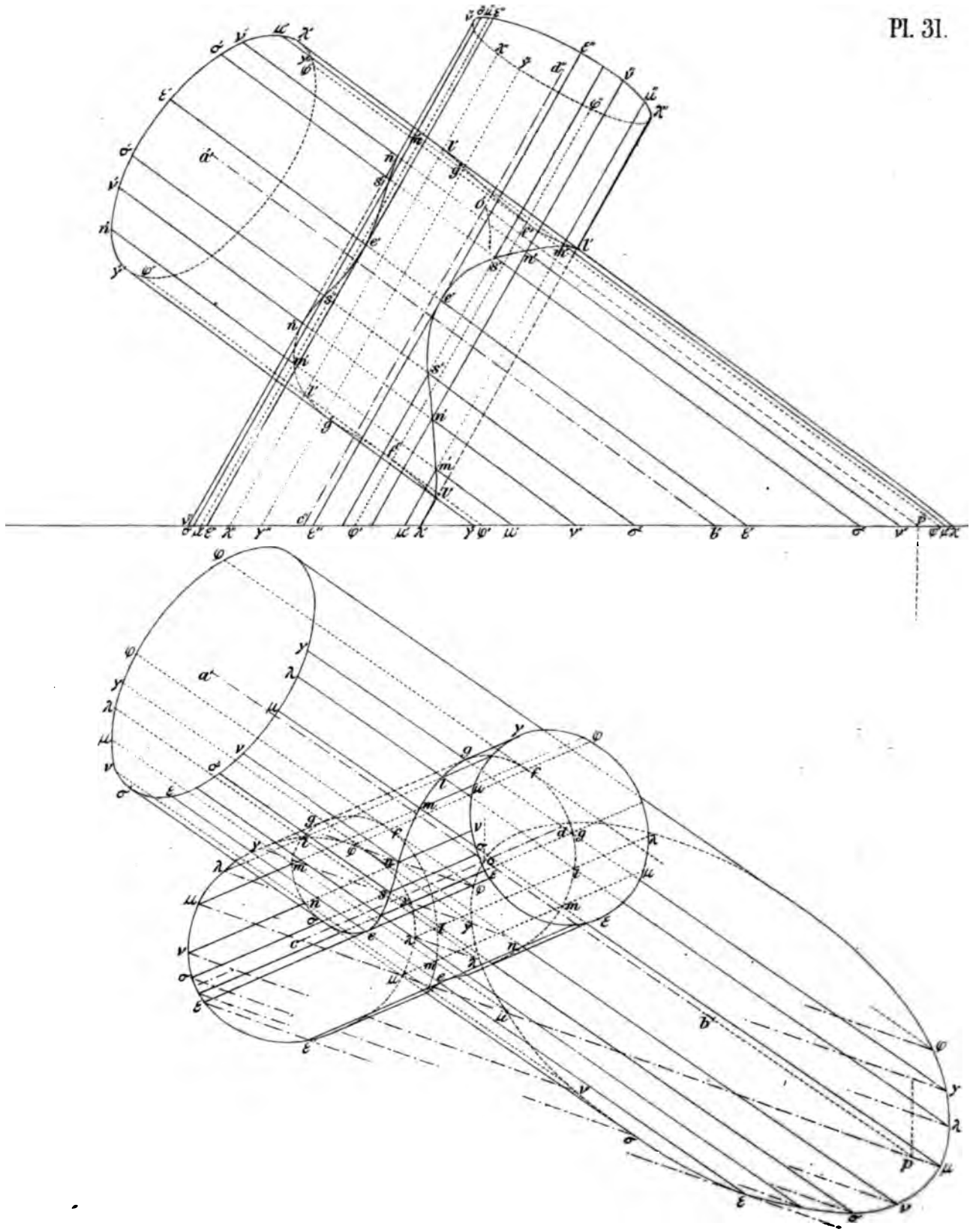




quired for finding the intersection are precisely the same as in the first figure, and need no further elucidation.

65. To find the intersection of two given interpenetrating cylinders.

(Plate XXXI.) Let A B, C D, be the axes of the given cylinders, which in the example do not meet. The general solution is, to suppose a series of auxiliary planes parallel to both axes, to cut the two cylinders. Each plane will cut each cylinder in two parallel straight lines. These two pairs of parallel straight lines will meet, forming a parallelogram, the four angular points of which will be points in the required curve of intersection.

First, find the traces of each cylinder $\lambda\, \nu\, \varepsilon$, or the curves in which the horizontal plane of projection cuts each cylinder; this is readily effected by means of another elevation on $a\, b$ and $c\, d$ as ground lines; the construction is not shown in the figure, to avoid confusion; and will be an easy problem for the student to think out himself. Next, to determine a plane parallel to each axis; take any point O in A B, and draw O P parallel to C D. Find the horizontal trace of O P in p, then the line joining $c\, p$ will be the horizontal trace of a plane containing A B, and parallel to C D. To distinguish the lines in each cylinder cut by the same plane, the same Greek letters have been used for their horizontal projections, their vertical projections being doubly accentuated in the cylinder A B. The four points of intersection common to these two pairs of lines are all denoted by the same corresponding English letter. Thus, the auxiliary cutting plane $c\, p$ cuts the cylinder with axis $(a\, b,\ a'\, b')$ in the two straight lines denoted by $(\mu\, \mu,\ \mu''\, \mu'')$ and the other cylinder in the two straight lines $(\mu\, \mu,\ \mu'\, \mu')$; the four points of intersection being all denoted by $(m,\ m')$. By taking the planes sufficiently close, any number of points may be found, and the curve drawn in with the hand; but there are certain special points which should first be found, which are the points of contact to certain known tangents, which thus give the direction of the curve at those points, and thus render its delineation more accurate. The plane $\phi\, \phi\, \phi$ for instance touches the smaller cylinder along the line $(\phi\, \phi,\ \phi''\, \phi'')$ and cuts the larger cylinder along the two generating lines $(\phi\, \phi,\ \phi'\, \phi')$. These last lines will, in each projection, be tangents to the projections of the curve at the two points of intersection (f, f'). Similarly, for the cutting plane $\varepsilon\, \varepsilon\, \varepsilon$, which touches the larger cylinder along the generating line $(\varepsilon\, \varepsilon,\ \varepsilon'\, \varepsilon')$. We have thus four points of contact, and four leading tangents in each projection. The other tangents, to which the points of contact are to be found, are, for the vertical projection, the four contour lines $\nu''\, \nu''$, $\lambda''\, \lambda''$, $\gamma'\, \gamma'$,

$\lambda' \lambda'$ representing each cylinder, and for the horizontal projection, the three contour lines $\sigma\sigma$, $\gamma\gamma$, $\varepsilon\varepsilon$. As many more intermediate points may be added and the projections completed.

66. To determine the curves of interpenetration of two given cones.

(Plate XXXII.) In this case, the auxiliary cutting planes contain the line joining the vertices of the cones. Each plane will thus cut the cones in two generating lines, and the intersections of these lines will be points common to both cones, and therefore points in the required curve.

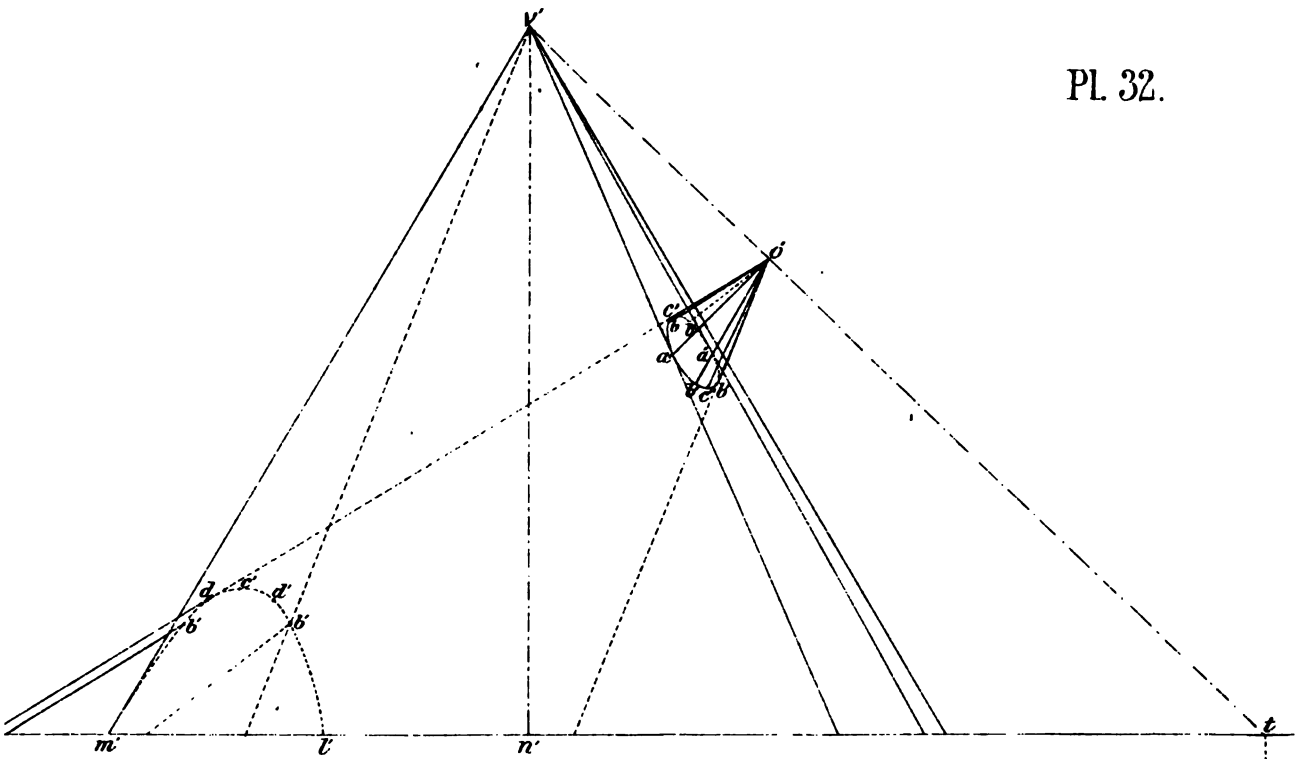
As in the last problem, there are certain guiding points which should first be found. Thus, in the figure, after having found the traces of the cones (67), and the horizontal trace of the line V O joining the vertices; draw the plane $a\,a$ touching the oblique cone along the generating line $(a\,o,\ a'\,o')$ and cutting the right cone in the lines $(\nu\,a,\ \nu'\,a')$; the intersections of these lines $(a,\ a')$ will give two points in the curve of intersection, and the generating lines $\nu\,a$, $\nu'\,a'$ will be the tangents to the curve at those points; the other special points are those to which the contour lines in each projection are the tangents, as shown in the drawing. As the figure is symmetrical with respect to the central line $r\,s\,p$, it will be useful also to determine the highest and lowest points in the curve. This is found by constructing a section of the two cones, made by the vertical plane $r\,s\,p$, and the points $(c,\ c')$, in which their contour lines intersect, will give the highest and lowest points required. This section is not shown in the figure, to avoid confusion. The tangents at the points $(c,\ c')$ will be horizontal.

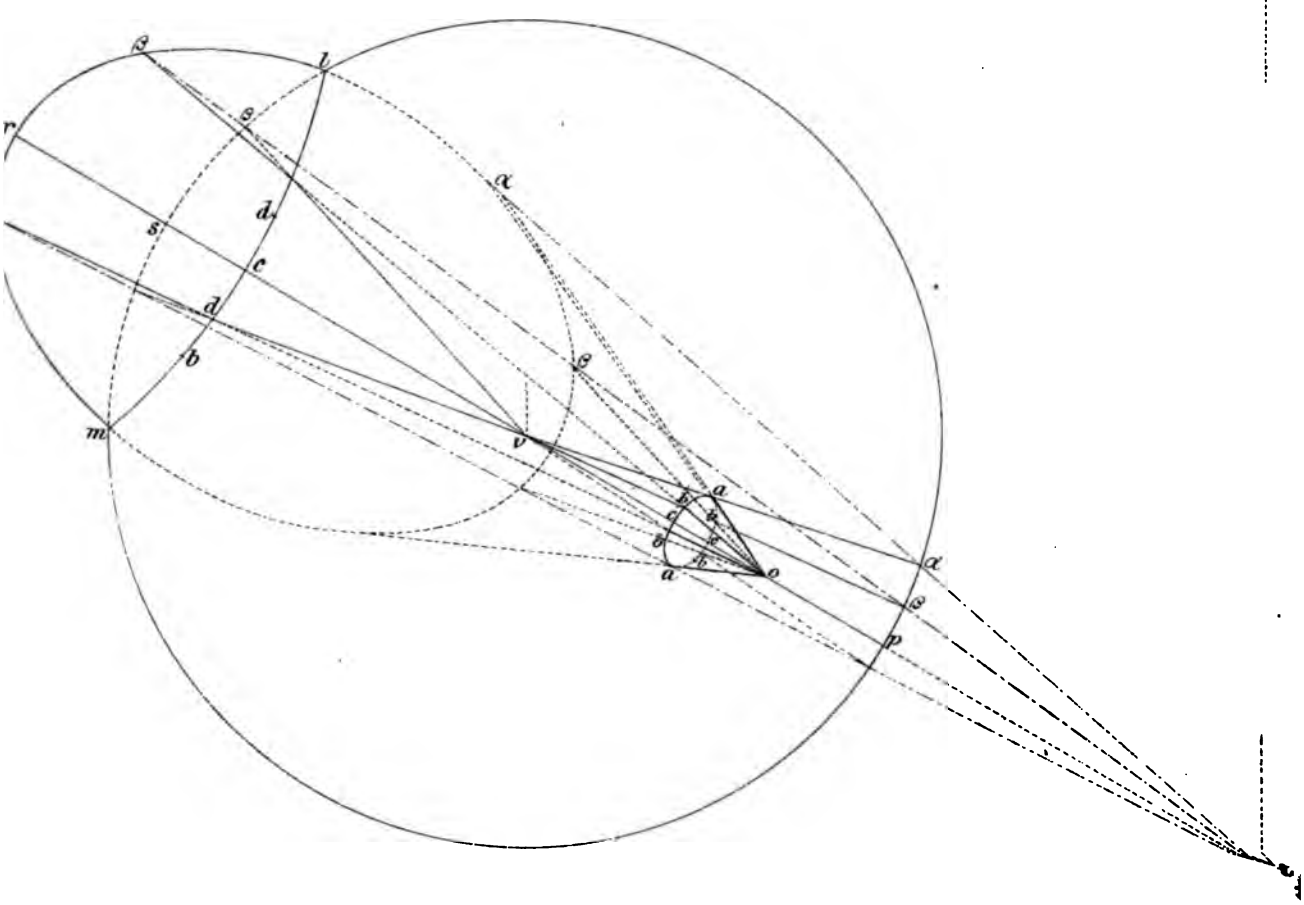
Plate XXXIII., gives a more general example of this problem, both cones being obliquely inclined to the horizontal plane. The method of finding all the principal points in the curve of interpenetration is sufficiently indicated by the construction lines in the drawing. The only difficulty that will probably be experienced both in this example and the preceding, is in finding the horizontal trace of the cone; as the special construction for determining this has been omitted to avoid confusing the figure, the method is shown separately in Plate XXXIV. The problem is the following:

67. Having given the plan of the axis of a right cone, its inclination to the horizontal plane, the height of the vertex, and the real magnitude of the vertical angle, to determine the horizontal projection of the cone and also its horizontal trace or the section made by the horizontal plane of projection.

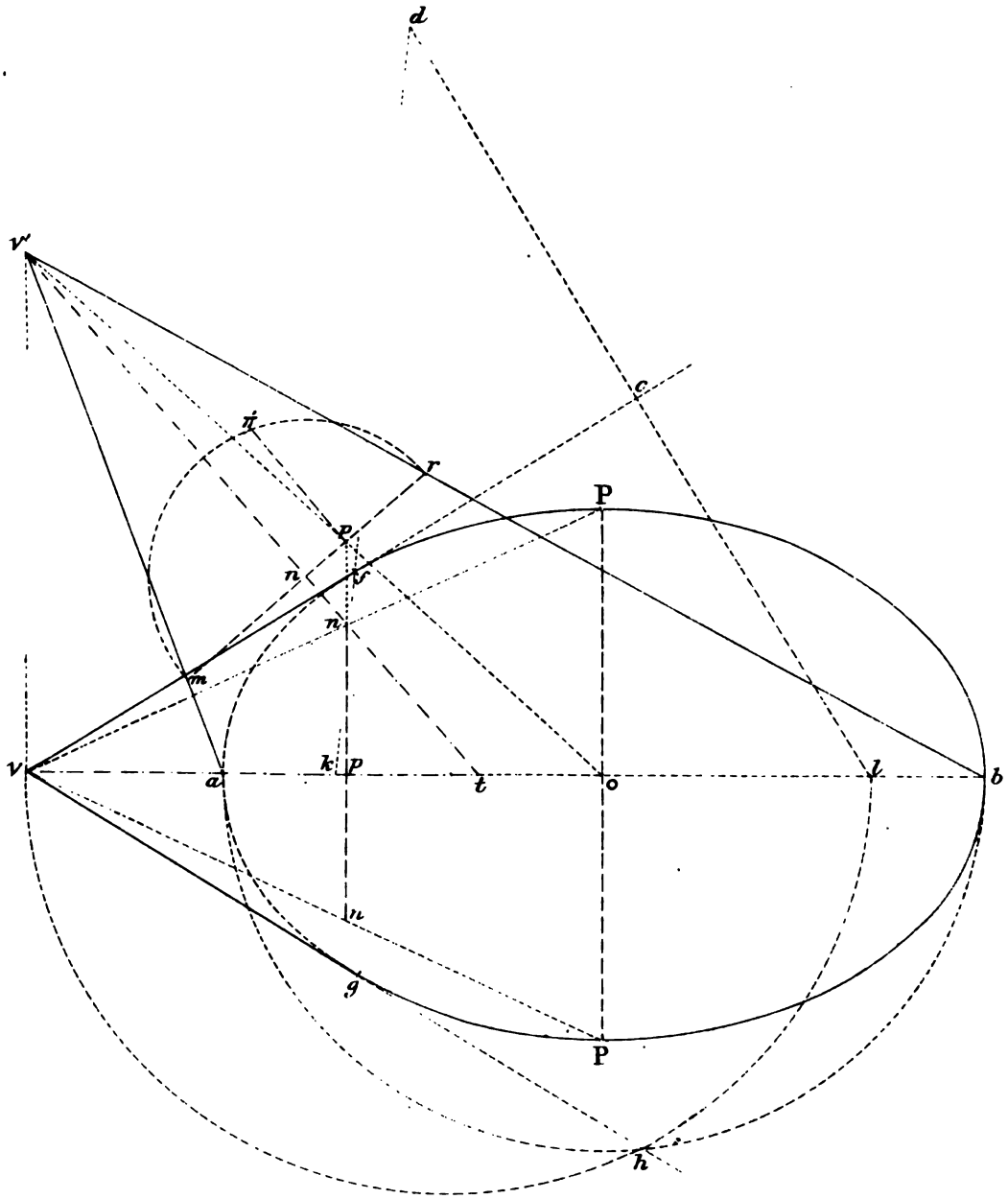
(Plate XXXIV.) Let $r\,t$ be the plan of the axis of cone, $v'\,t$ its vertical

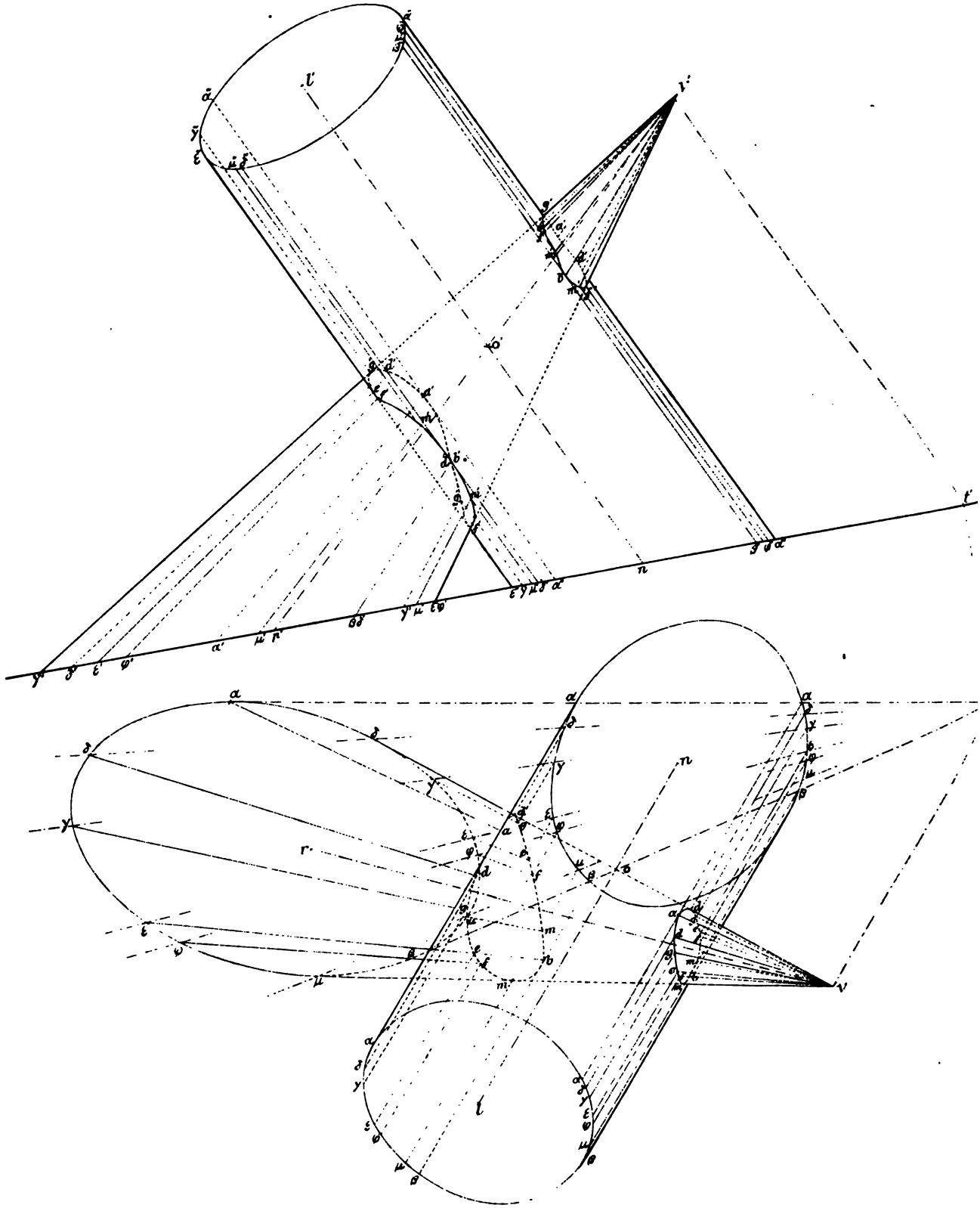
Pl. 32.





Pl. 34.





projection, taking $v\ t$ as ground line. Make the angles $t\ v'\ a$, $t\ v'\ b$ equal to half the given vertical angle of cone, then $a\ b$ will be the major axis of the ellipse formed by the intersection of the horizontal plane of projection with the cone or the trace required; it only remains, therefore, to determine the minor axis in order to complete the trace.

Bisect $a\ b$ by the perpendicular P o P, join $o\ v'$, draw $m\ n\ r$ at right angles to $v'\ t$ at any point n in $v'\ t$. This will represent the vertical projection of the circular section made by a plane cutting the cone perpendicular to the axis at the point n. Let $o\ v'$ intersect $m\ r$ in p'. Draw $p'\ p\ \pi$ at right angles to the ground line $r\ a\ b$. To determine the actual length of the horizontal ordinate projected vertically in p', suppose the circle $m\ p'\ r$ to be turned about its diameter $m\ r$ until it coincides with the vertical plane of projection in the circle $m\ \pi\ r$. Draw $p'\ \pi'$ at right angles to $m\ r$, and set this length from p to π on each side of $a\ b$. Join $v\ \pi$ and produce the line to cut o P in P; then o P will be the minor axis required, and the ellipse can be completed. From v draw the tangents $v\ f$, $v\ g$; this may be done by taking P as centre and $a\ o$ as radius and describing an arc cutting $a\ b$ in k and l the foci of the ellipse. On $a\ b$ describe a semicircle and on $v\ l$ describe a semicircle intersecting that on $a\ b$ in h, join $v\ h$, it will be the tangent required. It is often necessary to know the exact point of contact. To do this, draw $l\ c$ perpendicular to the tangent $v\ f$ produced, produce $l\ c$ to d, making $c\ d$ equal to $l\ c$, and join $d\ k$, cutting the tangent $v\ f$ in f; f is the point of contact required.

68. To find the interpenetration of a given cone and cylinder.

(Plate XXXV.) In this example the axis of the given cone V R meets the axis of the given cylinder L N in the point O. The auxiliary secant planes are assumed as passing through the vertex of the cone V and containing a line through V parallel to the axis of the cylinder L N. All these planes, since they pass through the vertex of the cone, will each cut the cone along two generating lines, and since they contain a line parallel to the axis of the cylinder, they will be parallel to the axis, and hence cut the cylinder along its generating lines. The figure sufficiently indicates the method of obtaining the projections of all the leading points in the curve of interpenetration as in the preceding problems.

69. To find the interpenetration of two given prisms.

(Plate XXXVI.) In the example, one of the prisms is square and the other hexagonal. First, find the traces of the two solids A B C D E F and

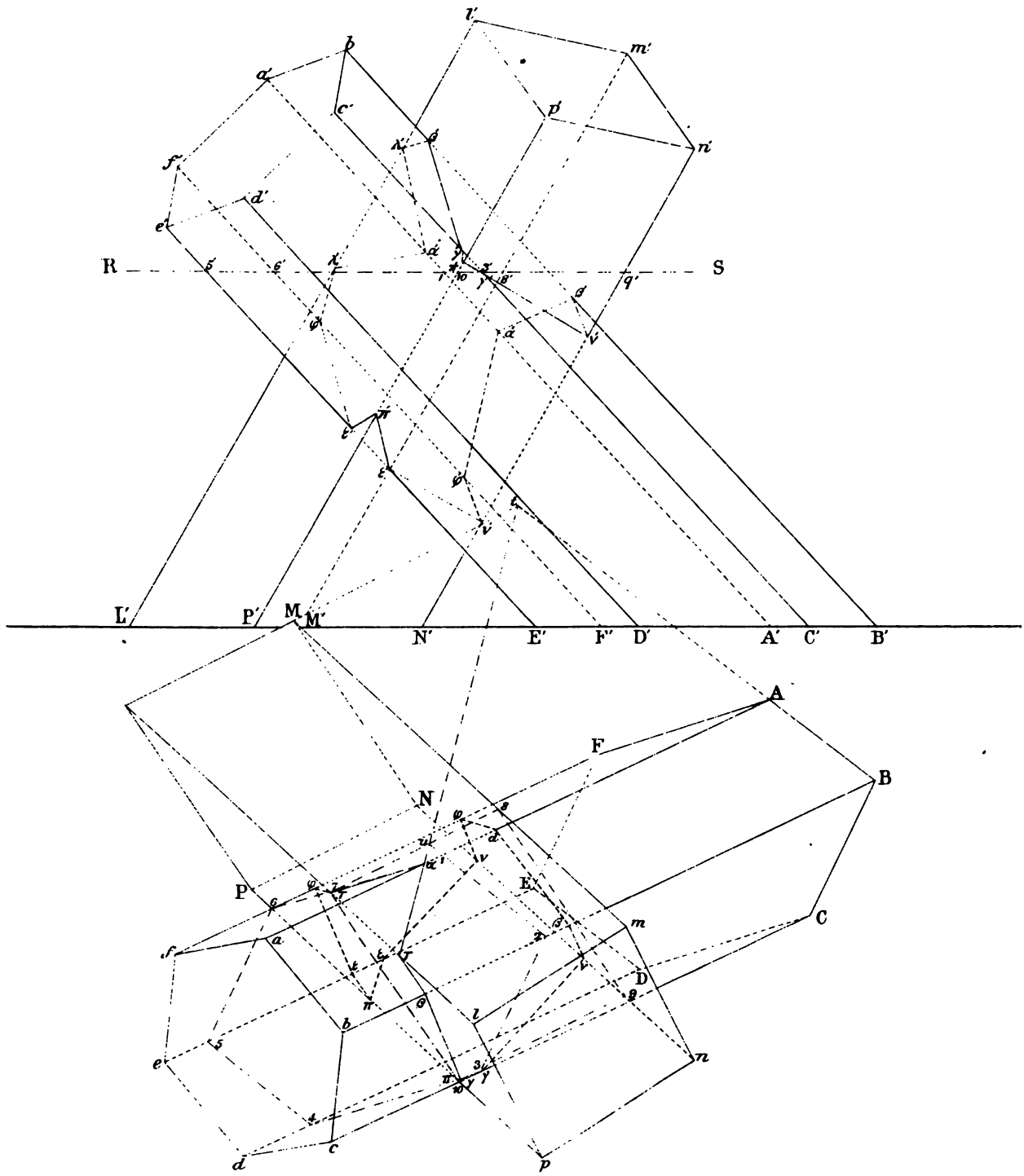
L M N P. Next, determine the horizontal projections of the intersections of any second horizontal plane as R S with each prism. 1, 2, 3, 4, 5, 6, is the section made by this plane with the hexagonal prism and 7, 8, 9, 10, the section it makes with the other. Now to find the line of intersection of any face of one prism with any face of the other, it will be sufficient to find two points in this line. Thus, to find the intersection of the face A B *a b* with the face L M *l m*, produce the horizontal trace of each plane A B, L M to meet in *t*, this will be one point in the line of intersection required; produce also the two corresponding horizontals 1, 2, and 7, 8, (lying in the same horizontal plane R S) to meet in *u*. Join *t u* and produce it to cut A *a* in *a* and L *l* in λ; *a* λ is the horizontal projection of the intersection required from which the vertical projection may now be deduced. Similarly, for all the intersecting faces. It is to be observed, that if one solid entirely pierces through a second, there will be two curves or polygons, one of entrance and the other of exit; but if they intersect only partially, part of one solid being within and part without the other, the intersecting curve or polygon will be a closed one or return upon itself as in the above example.

70. (Plate XXXVII.) This plate shows the intersection of a given pyramid with a given prism. The right section of the pyramid is a regular 12-sided polygon and that of the prism an equilateral triangle.

Since the pyramid in this example is assumed to rest on its vertex, the horizontal traces of the planes of all its faces will necessarily pass through its vertex and be parallel to the corresponding horizontal line in each face made by any second auxiliary horizontal cutting plane as R S. The method of construction is precisely the same as that shown in the previous example.

71. The interpenetration of a given cylinder and sphere.

(Plate XXXVIII.) For the sake of simplicity, one of the co-ordinate planes of projection should be assumed as parallel to the axis of the cylinder, in this case the vertical co-ordinate plane has been so assumed, and the auxiliary section planes are vertical planes parallel to the axis of cylinder, and passing through points taken on the right section of the cylinder at equal distances as A B C D E. These vertical planes cut the cylinder in rectangles and the sphere in vertical circles parallel to the vertical plane of projection, and can therefore be readily projected, and the points common to the sections of the two surfaces by each vertical plane will be points on the curve of interpenetration required. It is to be noted that the vertical plane which touches the cylinder along the generating line A A, will cut the sphere in a circle which will


R
S
L′
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M
M′
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Fig. 1.

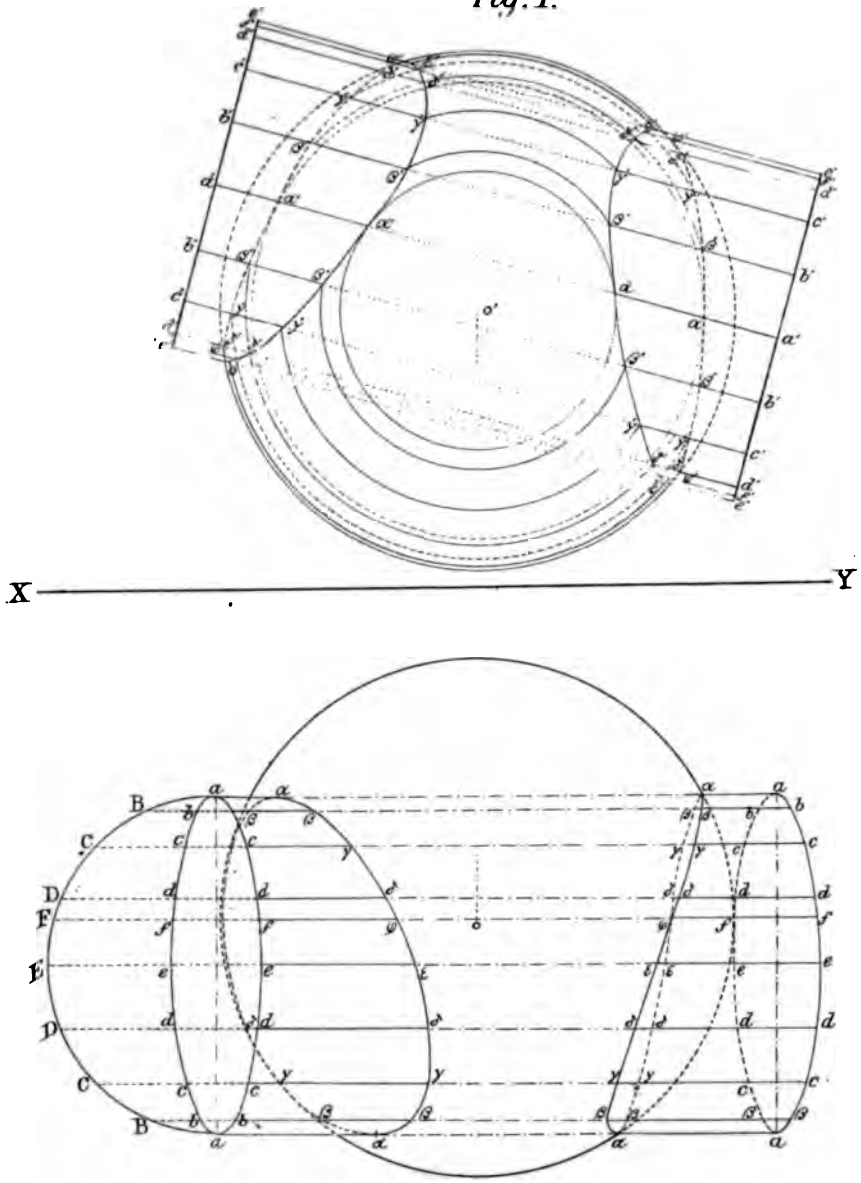


Fig. 2.

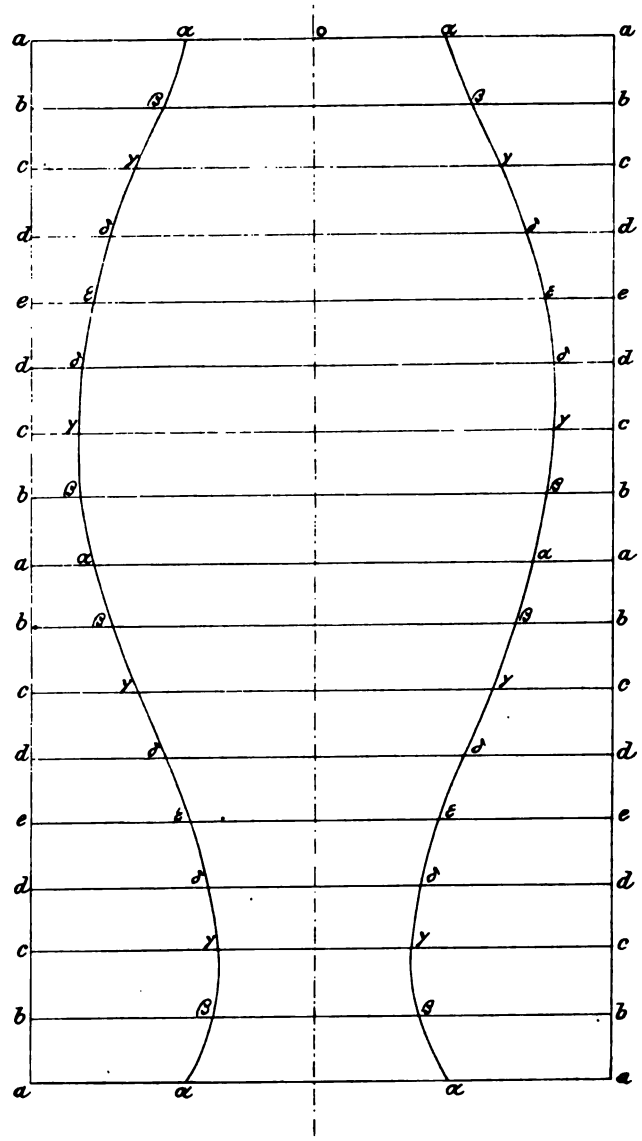
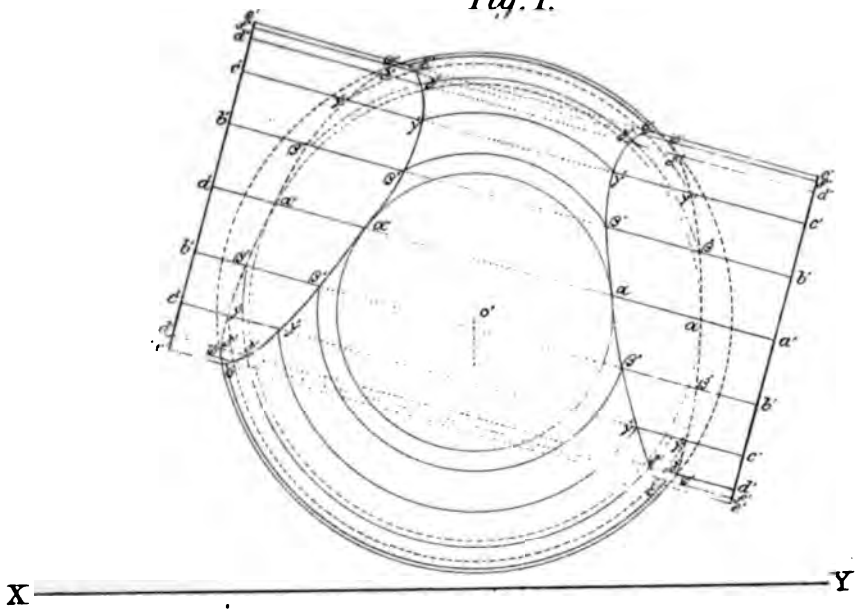


Fig. 1.



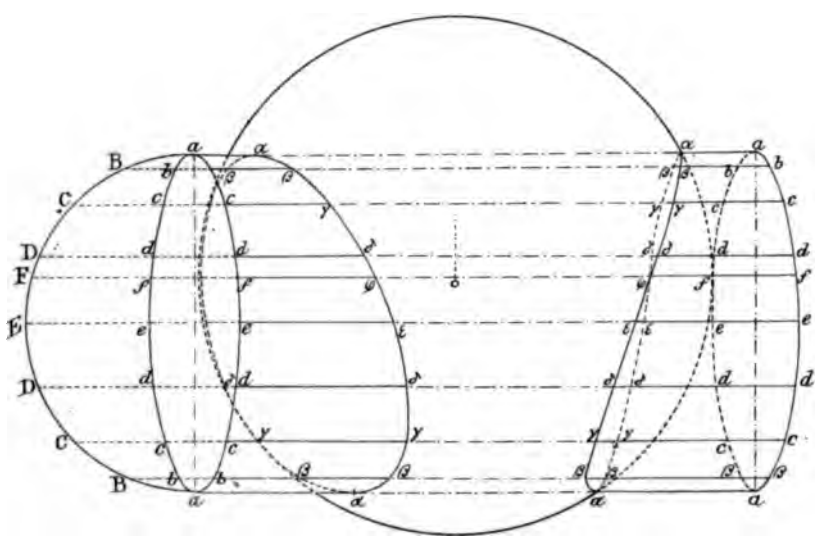
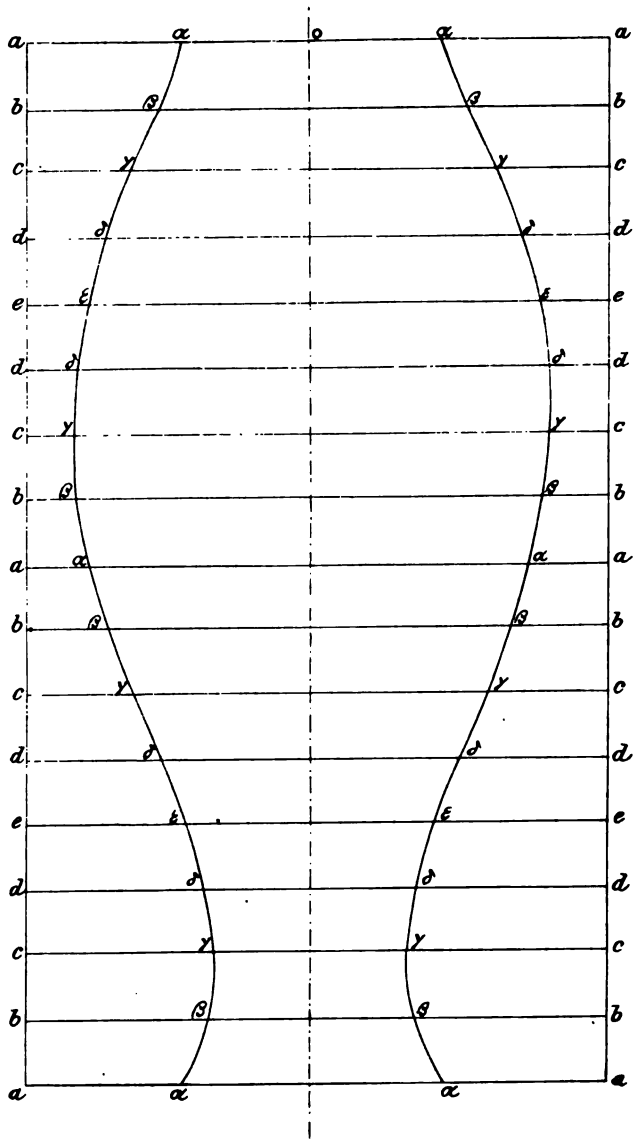


Fig. 2.



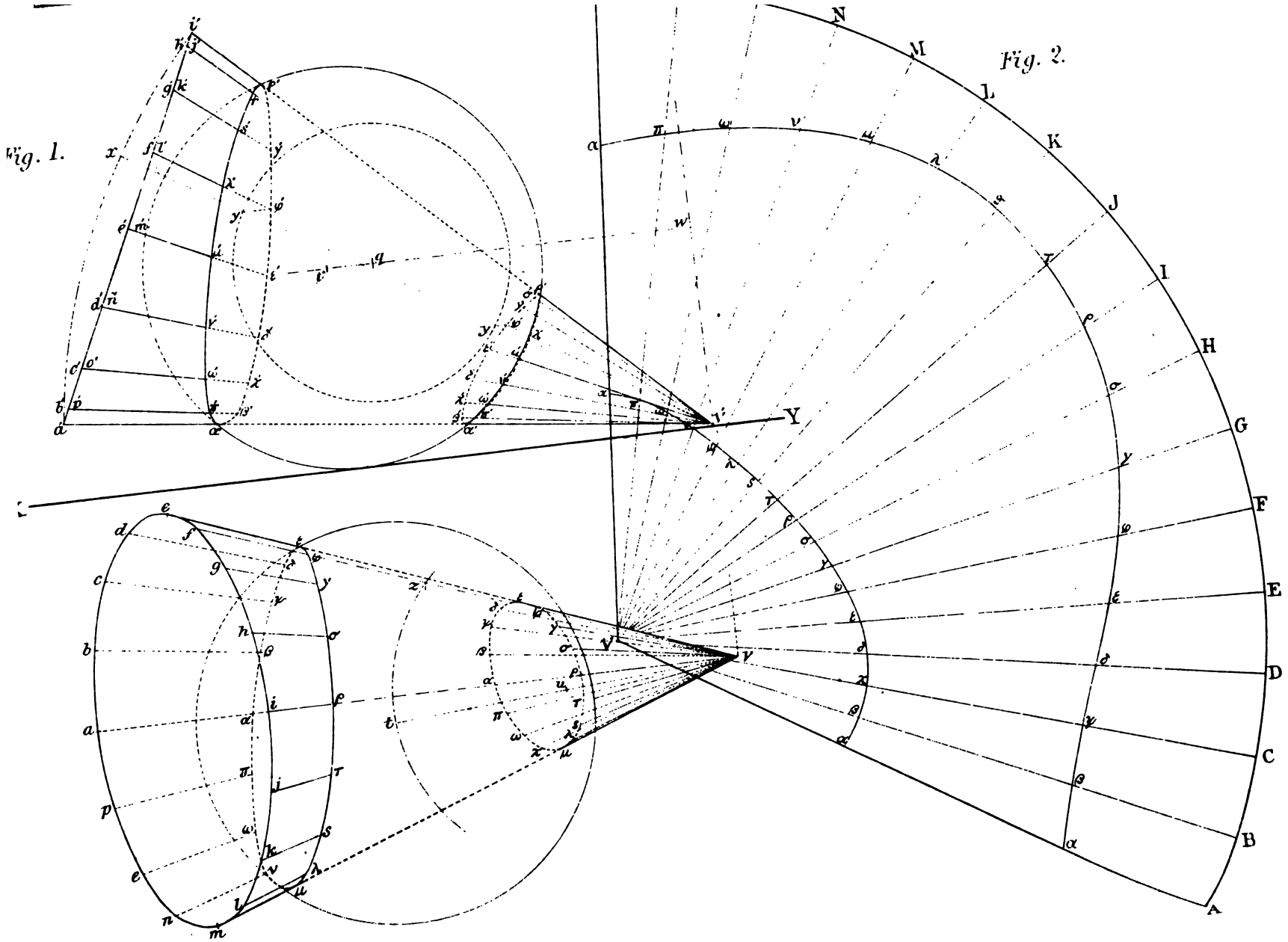


Fig. 1.
Fig. 2.
N
M
L
K
J
I
H
G
F
E
D
C
B
A
Y

have a common tangent with the curve of interpenetration at the point *a* where it cuts the line A A.

The plane F F passing through the centre of the sphere, gives the four points of contact f' f' f' f' where the vertical projection of the required curve touches the vertical contour of the sphere.

Fig. 2. shows the development of the cylinder, and of the two curves of entrance and exit; it will be seen that these curves are exactly symmetrical with respect to a central line $o\,o$.

72. The interpenetration of a right cone and sphere. (Plate XXXIX.) For the sake of convenience, the vertical plane of projection is assumed to be parallel to the axis of the cone. After dividing the base or any right section of the cone into any number of equal parts as in (a, a'), (b, b'), (c, c') &c., suppose vertical planes to pass through each of these points in succession and the vertex of the cone (r, r'). For example, take the vertical plane projected horizontally as the line $v\,f$; this plane will cut the cone in the line $(v\,f, v'\,f')$ (the other line of intersection is not shown) and the sphere in a vertical circle, the centre of which is the same level as the centre of the sphere. To exhibit this circle in the vertical projection, suppose the vertical plane containing it and the line $v\,f$, $v'\,f'$ to revolve about a vertical line passing through the vertex of the cone until it becomes parallel to the vertical plane of projection; the point (f, f') will describe a horizontal circle, projected vertically in the line $f'\,x$, and since when the line $(v\,f, v'\,f')$ is parallel to the vertical plane of projection, it will be projected vertically in its real length, the point x will lie on the circle described with v' as centre, and $v'\,a'$ or $r'\,i'$ as radius; then $x\,v'$ will be the vertical projection of the line $v\,f$, $v'\,f'$ in its new position. Again, the centre of the circular section made by the vertical plane and sphere, will describe a horizontal circle projected vertically in the line $q\,w$, drawn parallel to X Y through t', the vertical projection of the centre of the sphere; make $w\,q$ equal to $v\,z$, and with q as centre, and radius equal to the radius of the circular section derived from the plan, describe the circle $y\,y$, cutting $x\,r'$ in y, y. Lastly, suppose the vertical plane to resume its original position; the points of section y, y will describe the horizontal circles projected vertically in $y\,\phi'$, $y\,\phi'$, cutting $f'\,r'$ in ϕ' and ϕ', these points are the vertical projections of the points in which the line F V cuts the sphere, the places of which may now be deduced. Similarly with all the other points.

Fig. 2 shows the development of the two curves of interpenetration.

73. When the axes of two intersecting surfaces of revolution can be produced

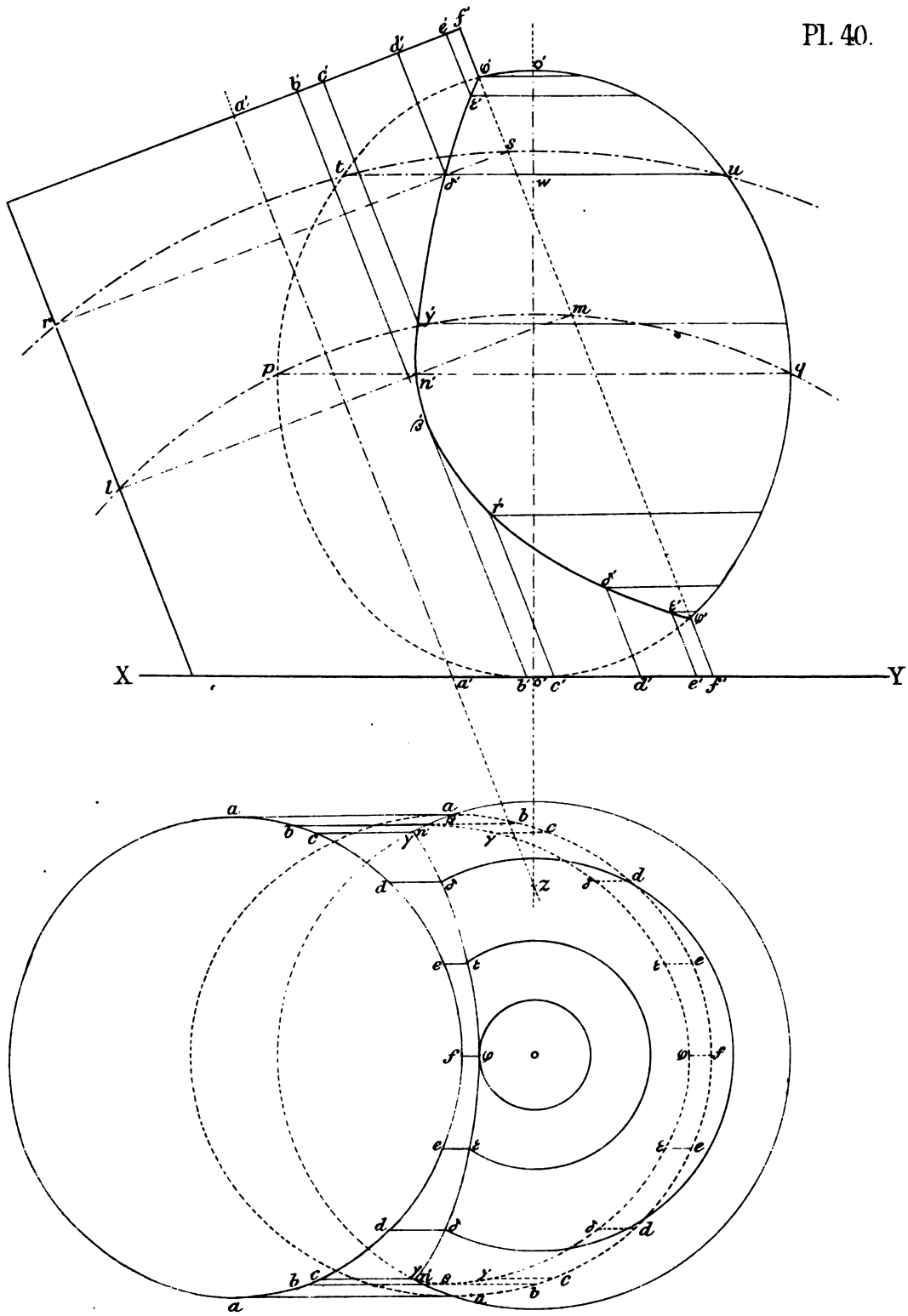
to meet, it is generally very convenient to form the auxiliary sections by a series of concentric spheres, their common centre being the point where the two axes of revolution of the surfaces meet.

(Plate XL.) This plate shows the interpenetration of a cylinder and ellipsoid, their axes of revolution intersecting in the point Z. The vertical plane of projection is assumed to be parallel to the plane containing the axes of revolution; hence the vertical projections of the circles, in which each sphere cuts the two surfaces, will be straight lines, perpendicular to their respective axes. The point (δ') in which these lines ($r\,s$ and $t\,u$) intersect, will be the vertical projection of two points in the curve of interpenetration required; their plans may be readily deduced, as lying on the plane of the horizontal circle described with centre o and radius $\omega\,t$. Similarly for all the other spheres. It is important to arrive at the intersections caused by the sphere cutting the ellipsoid in its greatest horizontal circle $p\,q$, as they will give the two points n, n in the plan, separating the visible part of the line of interpenetration from the invisible.

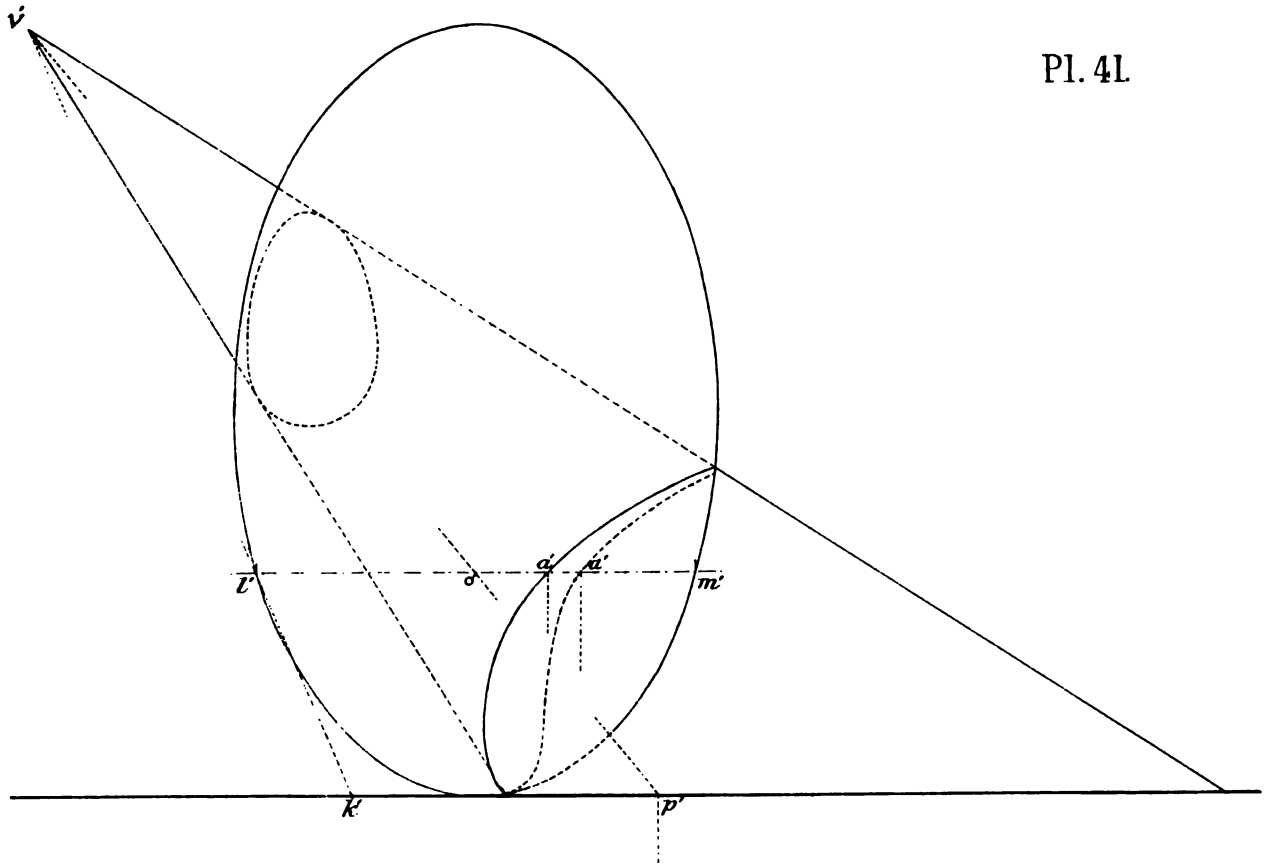
This method may be applied to all surfaces of revolution in which the axes about which the solids revolve, may be produced to meet.

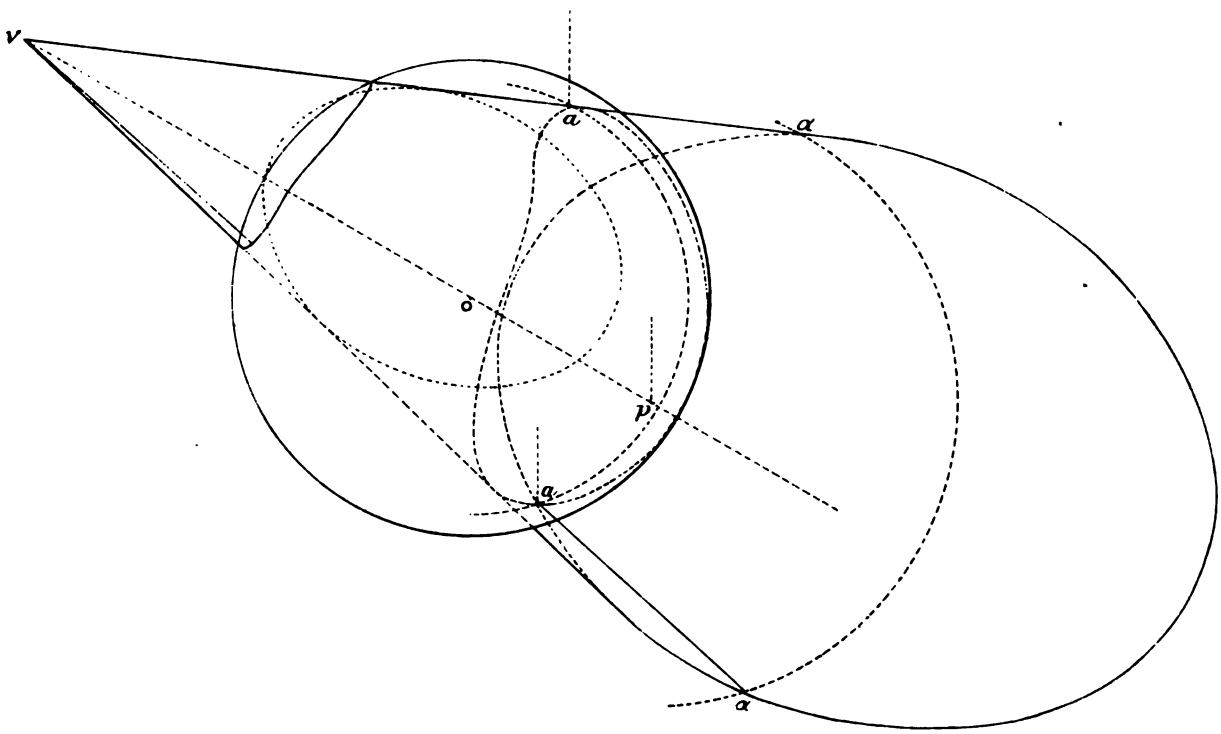
74. It only remains to describe an ingenious method of finding the interpenetration of two surfaces of revolution, when the horizontal section of one is circular and that of the other elliptical. It would be very laborious to construct the points of intersection by a series of auxiliary horizontal section-planes. Instead of this, the construction shown in Plate XLI. may be made use of. The solids of revolution are a vertical ellipsoid and an oblique cone, the axes of which do not meet. Suppose any horizontal plane to cut the ellipsoid in the circle L M, and let another cone be constructed, having the same vertex V as the given cone, and containing the horizontal circle L M. The trace of this second auxiliary cone will be the circle described with radius $p'\,k'$ and centre p, and ($a\,v$, $a'v'$) the lines in which the two cones intersect, cutting the circular section of the auxiliary cone in the points (a, a'); and since this circular section is also common to the ellipsoid, the points (a, a') must be common to the two given surfaces, and hence points in the curve of interpenetration required. By repeating this simple process, in which only the circle and straight line are employed, as many other points may be found as may be necessary.

If the second given solid had been an oblique cylinder instead of a cone, the auxiliary surface would be a cylinder having a parallel axis, and containing a horizontal section of the ellipsoid. We leave this example as a study for the student.

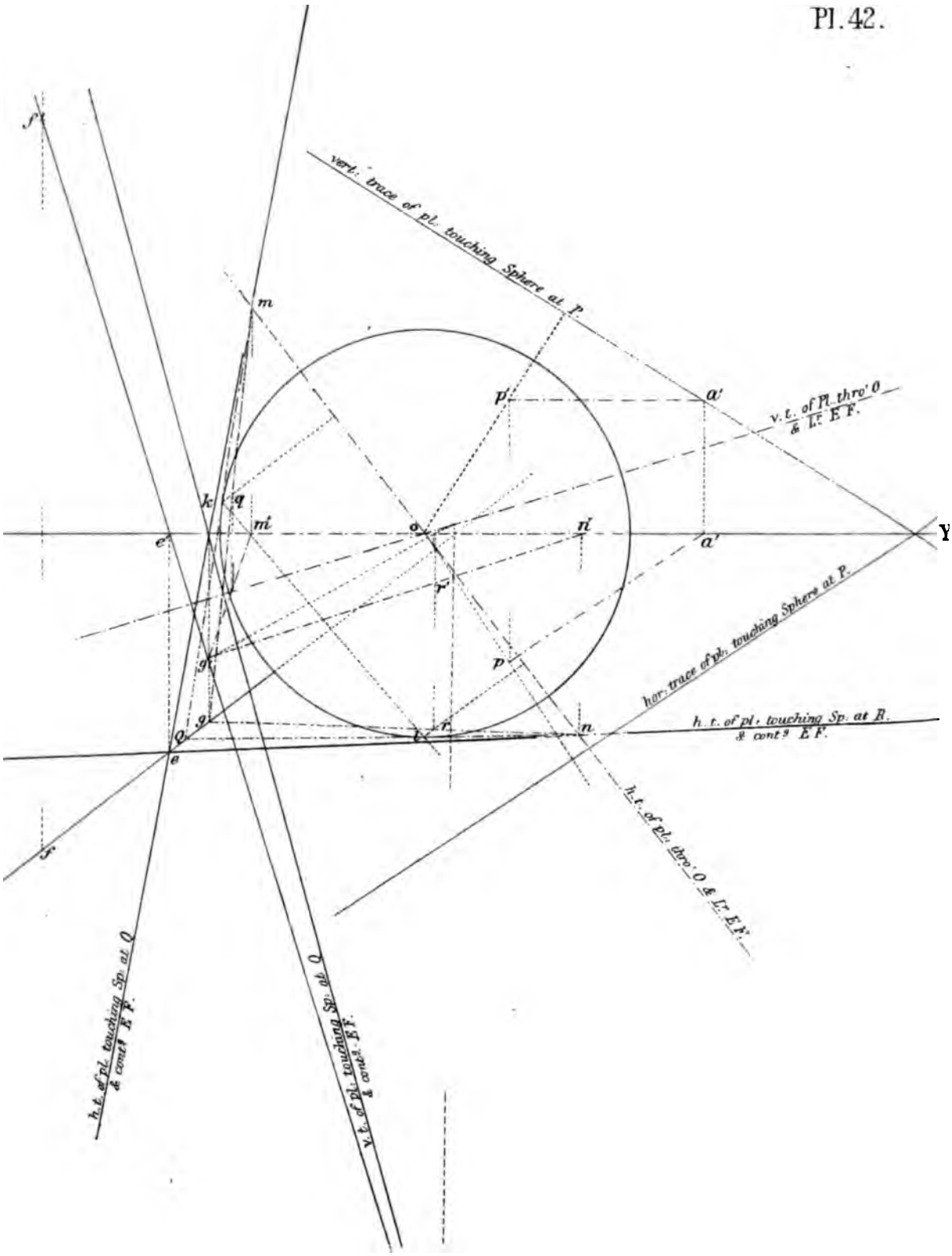

X
Y

Pl. 41.





Pl. 42.


vert: trace of pl: touching Sphere at P.
v. t. of Pl. thro' O & Lr. E F.
hor: trace of pl: touching Sphere at P.
h. t. of pl: touching Sp. at R. & cont.g E F.
h. t. of pl: thro' O & Lr. E F.
h.t. of pl. touching Sp. at Q & cont.g E F.
v.t. of pl. touching Sp. at Q & cont.g E F.
Y

CHAPTER VI.

ON TANGENT PLANES AND SURFACES IN CONTACT.

75. *Definition.*—A plane is said to touch a given surface at a given point when it contains the tangents to all curves that can be drawn on the surface at that point.

76. *Definition.*—A normal is a line drawn through the point of contact perpendicular to the tangent plane.

77. *Problem.*—To draw a tangent plane to a given sphere, having given the point of contact. (Plate XLII.) For the sake of simplicity the planes of projection are supposed to pass through the centre of the sphere, so that the same contour serves for both projections. The given point is (p, p') and O P is the normal. Hence a plane determined as containing P and perpendicular to O P will be the tangent plane required.

78. To draw a tangent plane to a given sphere and containing a given line.

In general there are two solutions to this problem; the special case in which only one tangent plane can be drawn, being that where the given line touches the sphere. If the given line cut the sphere, the problem is clearly impossible.

(Plate XLII.) The planes of projection are taken as in the preceding problem, and E F is the given line. Draw a plane through O, the centre of the sphere, perpendicular to the given line E F, and cutting it in the point (g, g'). This plane will cut the sphere in a great circle and the points in which tangents from (g, g') touch this circle will be the points of contact of the required planes. To determine these points suppose the plane through O and perpendicular to E F to revolve about its horizontal trace $m\,o\,n$ until it becomes horizontal. The great circle in which it cuts the sphere will now assume the position of the great horizontal circle of the sphere, and the point (g, g') having been constructed in G, the tangents G $k\,m$, G $l\,n$ can be drawn. The points m and n are the horizontal traces of the two tangents, and remain fixed during the motion of the plane.

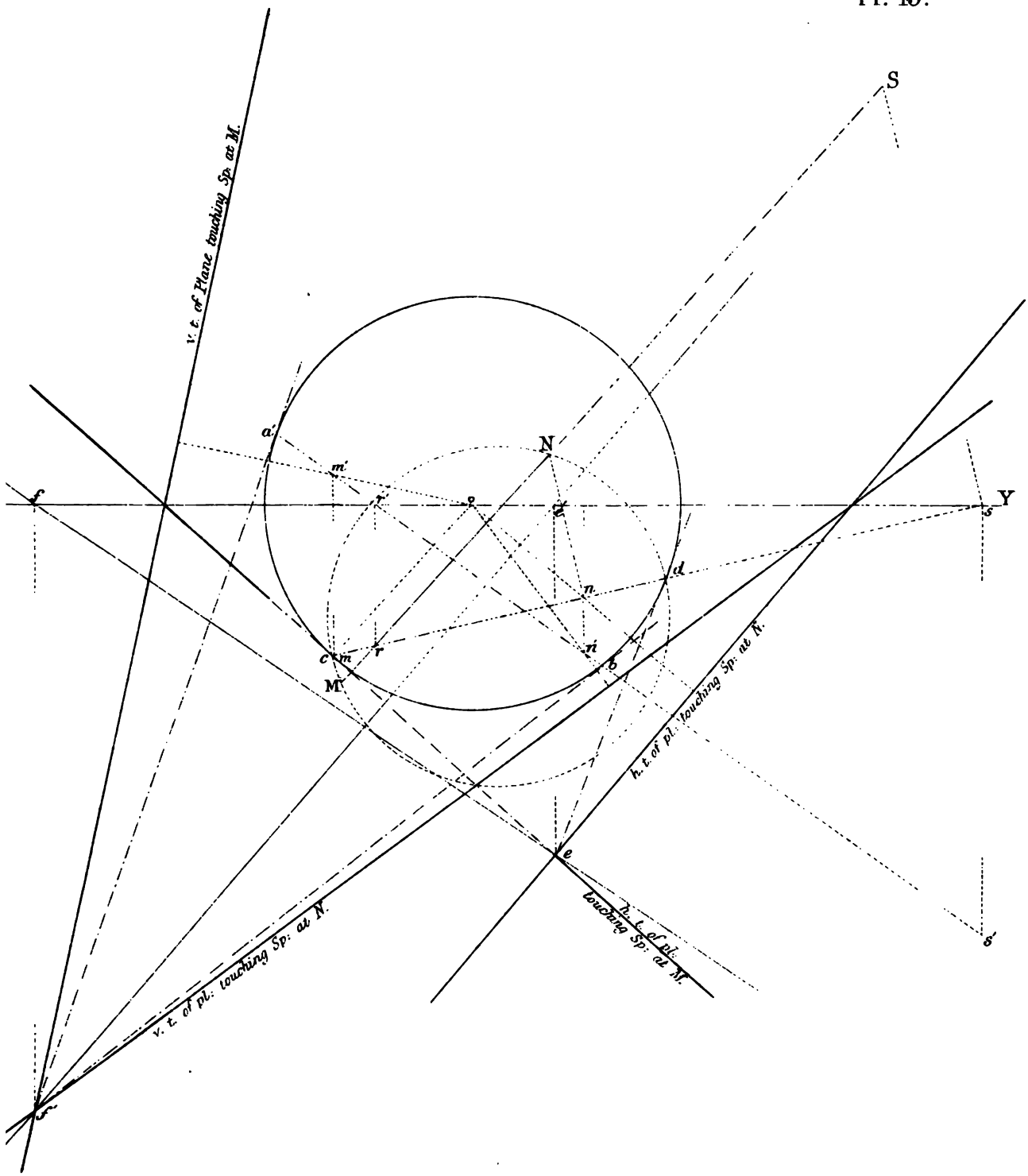
Moreover, since each of the tangent planes required must contain one of these tangents and the given line E F, the horizontal traces of these planes must contain the horizontal trace of E F and of the tangents respectively, hence $e\,m$ is the horizontal trace of one of the required planes, and $e\,n$ that of the other. When the plane containing the tangents assumes its original position, the point G returns to (g, g') and the plans of the tangents to $g\,m$, $g\,n$, cutting respectively the plans of the vertical circles $k\,g$, $l\,r$, described by the points of contact k, l, in q and r. These points lie on the lower surface of the sphere. The vertical traces of the required planes can now be determined as containing E F, and passing through the points of contact respectively. The vertical trace of the plane touching the sphere at R is not shown, as it lies without the limits of the paper.

79. *Another solution.*—(Plate XLIII.) Let e, the horizontal trace of the given line, be the vertex of a cone circumscribing the sphere; the plan of the circle of contact is the straight line $c\,d$, and as the points of contact of the required plane will necessarily lie in this circle, $c\,d$ is the horizontal projection of the chord joining the two points of contact. Similarly, if f', the vertical trace of the given line, be the vertex of a cone circumscribing the sphere, $a\,b$ will be the vertical projection of the chord joining the two points of contact. To determine the positions of these points let the vertical circle in which the cone with vertex e touches the sphere be turned about its horizontal diameter $c\,d$ until it becomes horizontal; the point r, which is the horizontal trace of the line $(c\,d, a\,b)$ joining the two points of contact, will lie on the axis of rotation $c\,d$, and will therefore not move, whereas the vertical trace of the same line (s, s') will move through a quarter of a circle s S, the radius of which is equal to $s\,s'$. Join S r and produce it to cut the constructed circle in M and N, the two constructed points of contact. By revolving the plane back into its original position the horizontal projections of the points of contact m and n can be found, and their vertical projections deduced. The traces of the tangent can now be readily found, as the horizontal traces will pass through the point e and be perpendicular to $o\,m$ and $o\,n$ respectively, and the vertical traces will pass through f' and be perpendicular to $o\,m'$ and $o\,n'$ respectively.

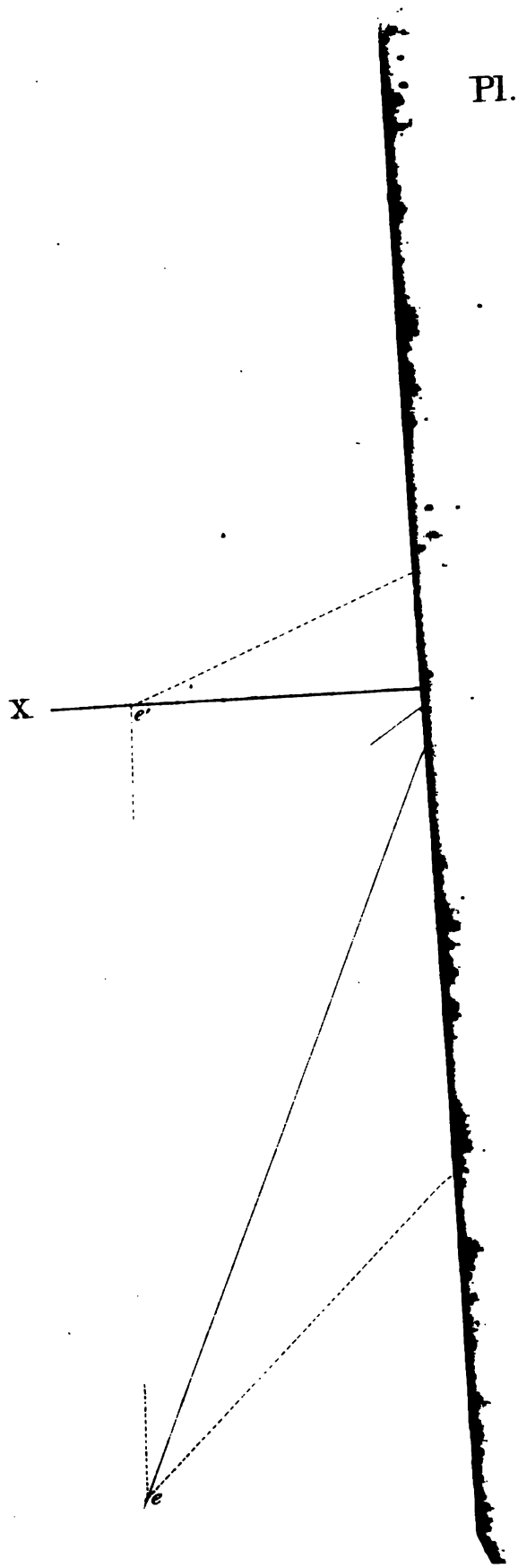
80. To draw a tangent plane (parallel to a given line) to a given cone and cylinder.

(Plate XLIV.) The above problem is identical with that of finding the shade and shadow of a given cone and cylinder, having given the direction of a

Pl. 43.


S
v. t. of Plane touching Sp. at M.
a'
m'
N
f
Y
s
d
n
c
m
r
M
b
h. t. of pl. touching Sp. at N.
e
h. t. of pl. touching Sp. at M.
v. t. of pl. touching Sp. at N.
s'

Pl. 44.



ray of light, and the rays being assumed to be parallel. In the projections given, the axis of the cone A B and that of the cylinder B C lie in the same straight line; and the right circular section of the cylinder, constructed by a plane through B perpendicular to B C, is taken as the base of the cone. First, to find the lines of demarcation of light and shade on the given cone. Through A, the vertex of the cone, draw the line A F parallel to the ray of light. Find the point G where this line cuts the plane of the base of cone. From G draw the two tangents G H and G K touching the circular base of the cone in H and K. The projections of these lines will evidently be themselves tangents to the ellipses which are the projections of the circular base of cone. Join A H, A I, these two lines will be the lines of contact of two tangent planes touching the cone and parallel to the inclination of the light. The horizontal traces of these tangent planes are readily found by drawing through H and I lines H N and I U parallel to the given ray; and the lines joining their horizontal traces F U and F N will be the horizontal traces of the tangent planes required and part of the shadow cast by the cone.

Next, to determine the lines of demarcation of light and shade on the given cylinder, or to find the tangent planes to the given cylinder, parallel to the given ray. Since the tangent planes must be parallel to the axis of the cylinder as well as the given ray, the two tangent planes will be parallel to the plane of the triangle A F C; therefore, their horizontal traces will be the two lines $q\,l$, and $t\,j$ drawn parallel to $f\,c$ and touching the horizontal trace of the cylinder in l and m; draw L M and J K, which will be the lines of contact required, and the space enclosed between the lines $q\,l$ and $t\,j$ will be the shadow cast by the cylinder. Between the lines of contact in the cone and cylinder there remains a space H M and I K the shadow of which may be found by taking several points such as R, S, and through these drawing lines parallel to the given ray; the horizontal traces of these lines will be points in the shadow required, and the curve may be drawn through them with the hand; it is necessary to observe in drawing these short curves that the lines, as $f\,n$ and $q\,l$, $f\,u$ and $t\,j$, will respectively touch the curves at the points n, q, and u, t.

81. Having given the projections $(a\,b,\ a'\,b')$ of the axis of revolution of an ellipsoid, and the magnitude of the radius of the equatorial circle, to determine the two projections of the solid.

Taking $a\,b$ as a ground line same level as A, project b in B making b B equal to the height of B above A. Then a B will be the vertical projection and

real length of the axis of revolution; draw O t' equal to the given radius and describe the ellipse $a\,t'\,c$ B described on a B as major axis and O t' as minor axis. This ellipse will represent the vertical projection of the solid on $a\,b$ as ground line. To complete the plan of the solid, it is necessary to determine the trace of a vertical cylinder enveloping the solid; this trace is also the locus of the intersections of the traces of all vertical tangent planes to the solid. Now, in all surfaces of revolution, the curve of contact of an enveloping cylinder is a plane curve and conjugate to the diameter which is parallel to the axis of the cylinder.

Thus draw O c perpendicular to $a\,b$, then the line $d'\,e'$ drawn through O parallel to the tangent at c will be the vertical projection of the curve of contact required. The horizontal projection of this curve will also be that of the solid, and can be now readily drawn as we know the two axes $d\,e$ and $o\,w$ equal to O t'.

The vertical contour or projection of the solid is derived in a precisely similar way, and is sufficiently indicated in the figure.

82. Having given the projections of an ellipsoid in any position, and the plan of a point on its upper surface, to determine the tangent plane at this point.

(Plate XLV.)—Having determined the projections of the solid as in the last problem, let p be the plan of the point; to determine the vertical projection of this point, construct the vertical plane containing the point, about the horizontal line $g\,h$ (projected vertically in the point n) and its section with the solid. Now, since the plane $d'\,e'$ is conjugate to the diameter O c, it will bisect all parallel chords as $n\,k$; hence the semi-major axis of the elliptical section made by the vertical cutting plane, will, when constructed, be the straight line $i\,m$ equal to $n\,k$; and $i\,g$ is the semi-minor axis of the same section. Draw $p\,q$ cutting this elliptic section in q; then $p\,q$ is the height of P above the horizontal line $g\,h$; make $n\,p'$ equal to $p\,q$, and p' will be the vertical projection of the point P required, on the projection having $a\,b$ as ground line; the distance $i\,p'$ above the height of a', will be the distance to which $s\,p'$ in the original plane of projection must be made equal.

Returning to our projection on $a\,b$, draw $p'\,l'$ at right angles to a B and produce it to meet the ellipse $a\,c$ B in P, and draw P r', a normal to the ellipse $a\,c$ B meeting the axis in r'; and since in surfaces of revolution the normals to all points in each circle of rotation, meet the axis in the same point, $p'\,r'$ will be the vertical projection of the normal at the point P, from which the plan $p\,r$ can

be deduced. The tangent plane is not shown in the plate, but can be easily found as containing P and perpendicular to P R the normal.

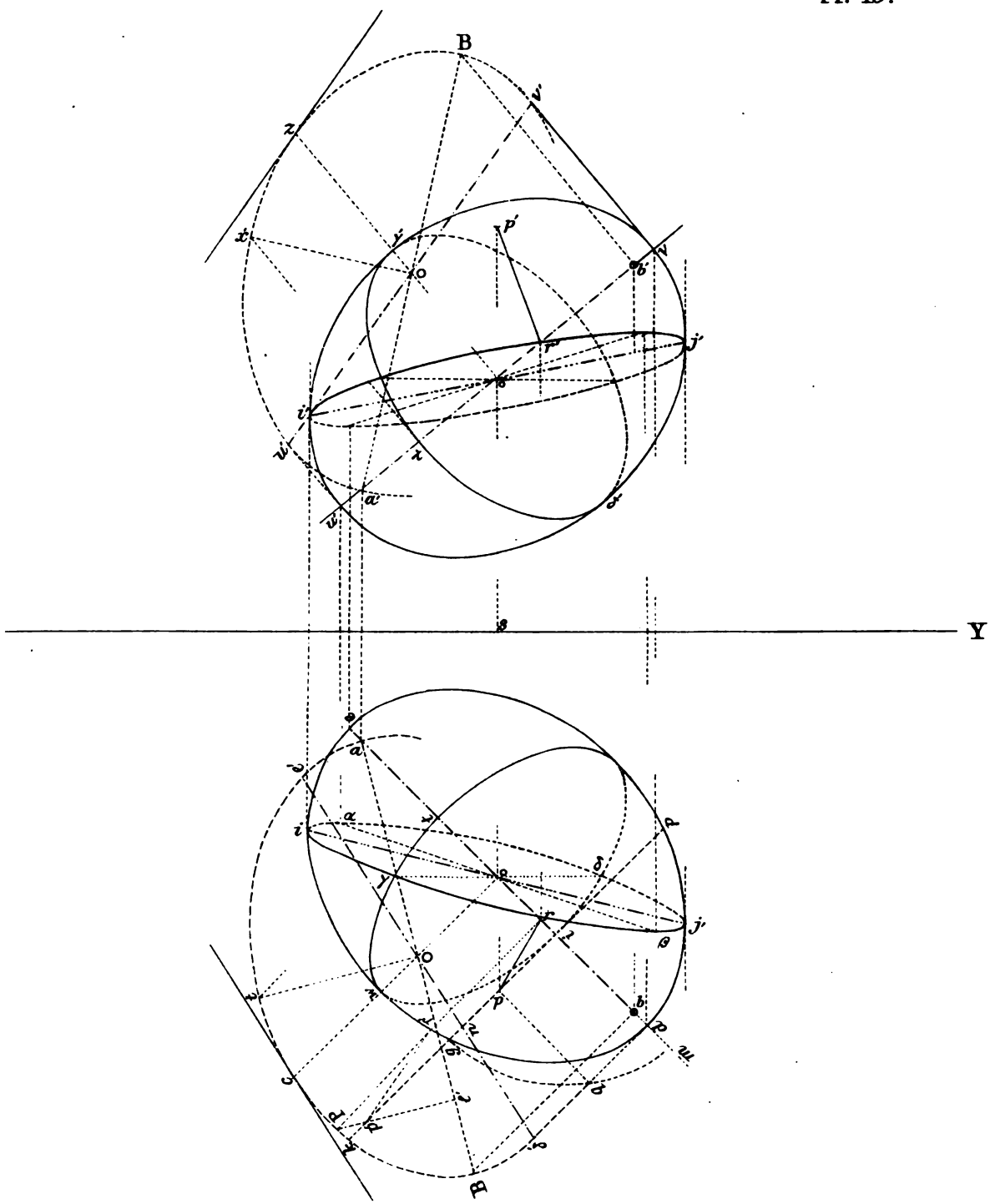
83. To find the plan of the curve of contact in which the circumscribing horizontal projecting cylinder touches the surface of revolution and of which the vertical contour of the surface is the vertical projection; also, to find the vertical projection of the curve of contact in which the circumscribing vertical projecting cylinder touches the surface of revolution and of which the horizontal contour of the surface is the horizontal projection.

First, the major axis of the plan of the first curve required and also that of the vertical projection of the second, will be the plan and elevation of the same straight line, formed by joining the points of contact of the two tangent planes to the surface, which are perpendicular to both planes of projection as $i\,j$, $i'\,j'$; another diameter of the plan of the first curve is found by projecting $\gamma'\,\delta'$ on the equatorial circle, by drawing the line $\gamma\,o\,\delta$ parallel to the ground line, and its conjugate diameter $\alpha\,\beta$, which is the plan of $u\,v$, by taking the distances of u and v from the vertical plane of projection, known by the auxiliary projection on $u\,v\,a'$ ground line same level as a'. Similarly for the second curve.

84. To find the curve of contact of a cylinder circumscribing a given surface of revolution and parallel to a given straight line.

The axis of the given surface is assumed to be parallel to the vertical plane of projection and inclined to the horizontal plane, as $a\,b$, $a'\,b'$; this position is sufficiently general, and should the axis be inclined to both planes, the construction must then be deduced from the projection on a plane assumed to be parallel to the axis. In the present case the vertical projection or contour of the given surface is the ellipse formed on $a'\,b'$ as major axis and $c'\,d'$ as minor axis. The horizontal projection or contour of the surface is derived from the vertical projection by the method shown in the preceding problem, viz. by finding the plan of the curve of contact made by the circumscribing vertical cylinder. This vertical cylinder will touch the surface in an ellipse projected vertically in the straight line $g'\,o'\,h'$ drawn parallel to the tangent at i', the point where the vertical line through $(O,\,o')$ cuts the surface; the points $g'\,h'$, projected on to the direction of $a\,b$ produced will give the major axis of the ellipse forming the required horizontal contour of the surface and $w\,o\,w = c'\,o'\,d'$ will be the minor axis, whence the ellipse can be drawn; the plan of the equatorial circle $w\,c\,w\,d$ projected from the vertical projection $c'\,o'\,d'$ can also be determined.

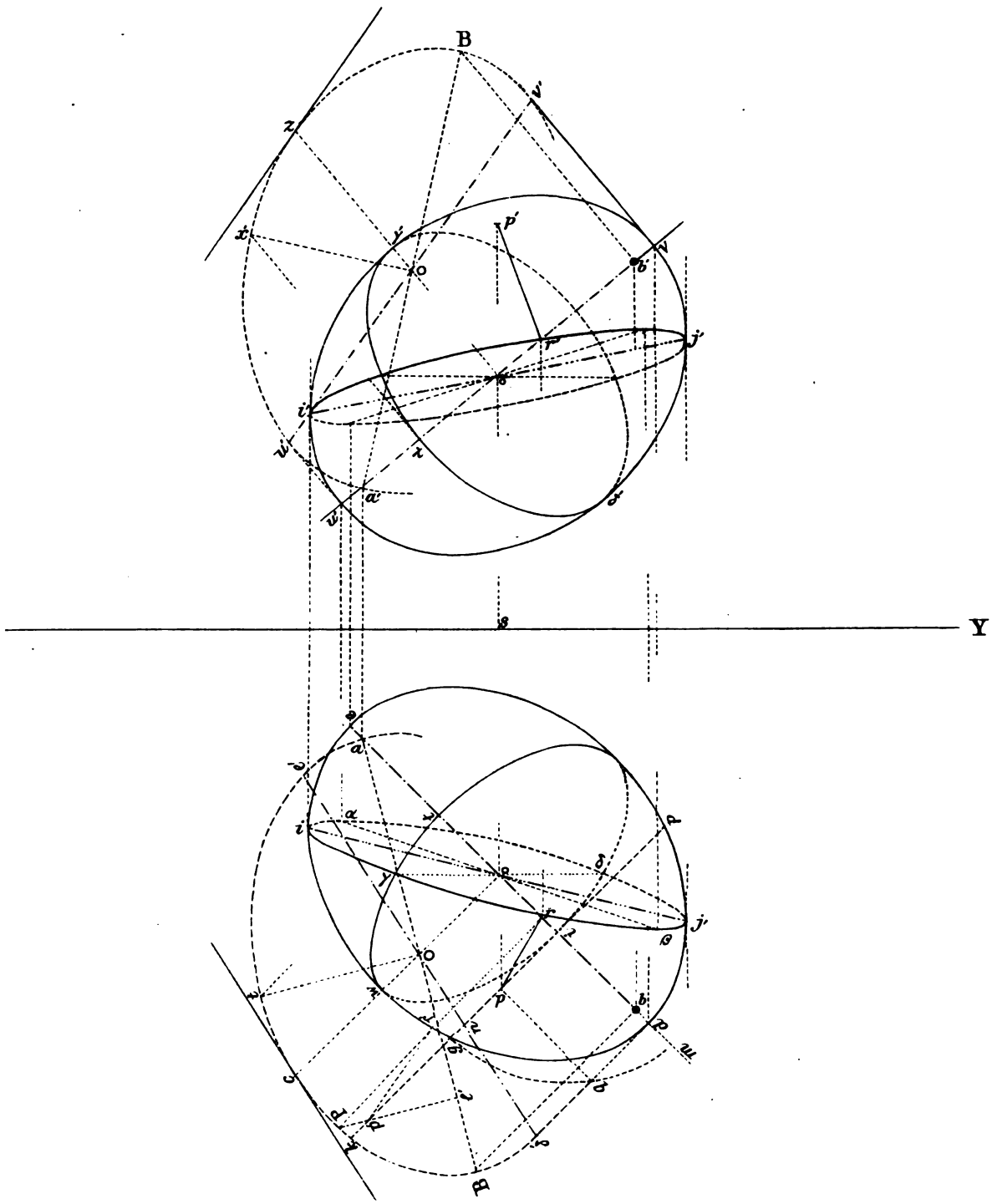
Pl. 45.



Through the centre of the surface o, o' draw the line $o\,j$, $o'\,j'$, parallel to the given inclination of the axis of the required cylinder. To determine the points in which this line cuts the given surface, suppose the vertical projecting plane containing this line $e\,f$ to cut the surface. The two points e, f, lying on the horizontal elliptical contour of the surface will be vertically projected in the points e', f', on the line $g'\,h'$; the two lines $(e\,f,\ e'\,f')$ and $(o\,i', o'\,i')$ will be two conjugate axes of the elliptical section caused by the vertical plane $e\,f$.

To exhibit this in the vertical plane, suppose the plane $e\,f$ to turn about the vertical line $(o,\ i'\ o'\ i')$ until it is parallel to the vertical plane of projection, the points (e, e') (f, f') will then be respectively projected in the points $(\varepsilon,\ \varepsilon')$ $(\phi,\ \phi')$, and the ellipse constructed, having for conjugate axes $\varepsilon'\,\phi'$ and $i'\,o'\,i'$, will be the elliptical section of the surface required; only one quadrant $i'\,l'\,\phi'$ is here shown; during this revolution, the axis of the required cylinder will take up the position $(o\,k,\ o'\,k')$ and $k'\,o'$ produced to meet the elliptical quadrant in (l, l') will give the point where the axis cuts the surface; by making $o\,m, o'\,m'$ equal respectively to $o\,l$, $o'\,l'$, we obtain the point $(m, m',)$ where the other extremity of the axis of the cylinder cuts the surface. Lastly, by revolving this vertical plane $e\,f$ back into its original position, carrying with it the points of intersection, we obtain the required projections of the points (n, n') $(p, p',)$ in which the axis of the cylinder cuts the surface of revolution.

Now the plane of the curve of contact required will pass through the centre of the surface (o, o') and be parallel to the tangent planes to the surface at the points (n, n') (p, p') in which the axis cuts this surface. Now the tangent plane at the point (p, p') will contain the tangents to all the curves that can be drawn on the surface through (p, p'). To find the tangent to the meridional curve through (p, p'); if this curve be supposed to revolve about $(a\,b,\ a'\,b')$ until it is parallel to the vertical plane of projection, its projection will coincide with that of the vertical contour, and the circle of revolution described by (p, p') will be projected vertically in the line $q'\,p'\,\pi$. Draw $\pi\,s'$ a tangent to the curve at π; the point (s, s') on $(a\,b,\ a'\,b')$, being a point on the axis of revolution, will remain fixed, therefore $(s\,p,\ s'\,p')$ will be the projections of the tangent to the meridional curve through (p, p'). The tangent at (p, p') to the circle of revolution $\pi\,q'$ can be readily found by constructing this circle by the quadrant described with q' as centre and $q'\,\pi$ as radius, and drawing $p'\,\pi$ at right angles to $\pi\,q'$; draw $\pi\,\tau$ a tangent to this circle, and the length $q'\,\tau$ will be the real length of the horizontal line $(q\,t,\ q')$; and $(p\,t,\ p'\,q')$ will be the projections of this tangent. These two tangents are at right angles to each other; if then lines be drawn through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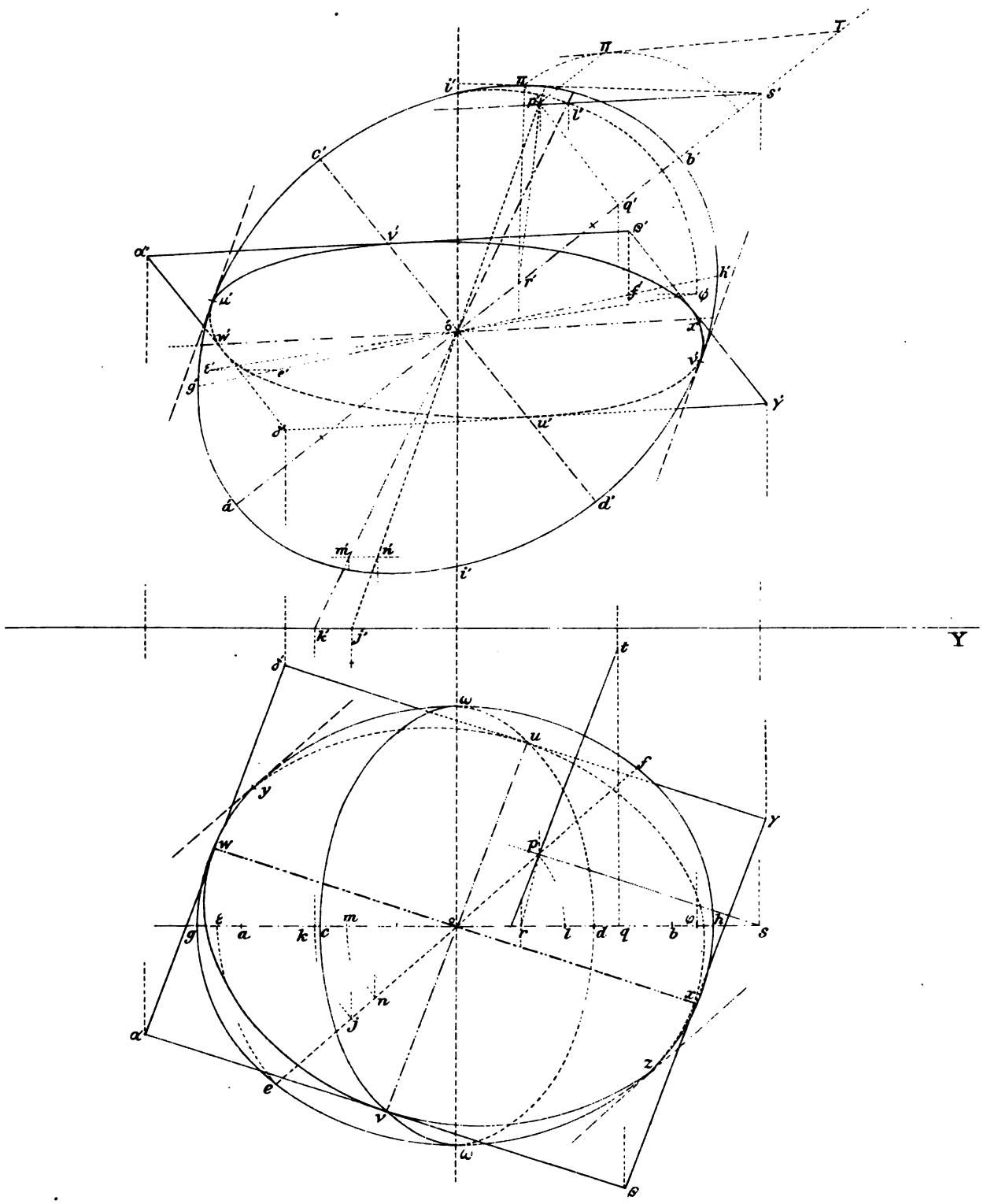

Y

the centre of the surface parallel to these tangents, and their points of intersection with the surface be determined, these lines will be conjugate axes of the ellipse forming the curve of contact. The line through (o, o') parallel to the tangent to the generating circle at (p, p') will cut the surface in the equatorial circle, hence its projections $u\ o\ v$, $u'\ o'\ v'$ can be readily determined. The intersections of the line $(w\ o\ x, w'\ o'\ x')$ with the surface are determined in a precisely similar manner to that which was explained for finding the intersections of the line $(e f, e' f')$. The parallelogram $(\alpha\ \beta\ \gamma\ \delta, \alpha'\ \beta'\ \gamma'\ \delta')$ is that which circumscribes the curve of contact required, and will help to draw the curve. The points in which the plan of the curve touches the horizontal contour, are determined by drawing tangents to this curve parallel to $e f$ touching the curve in y and z. Similarly, the points μ', ν' in which the vertical projection of the curve of contact touches the vertical contour of the surface, are the points in which the tangents, drawn parallel to vertical projection of the given direction of the axis of the cylinder, touch the curve.

85. To find the curve of contact of a cone circumscribing a given surface of revolution, and having a given vertex.

(Plate XLVII.) Let (v, v') be the given vertex of required cone. The given surface of revolution, in this case an ellipsoid, is determined under the same conditions as in the last problem, having its axis of revolution $(a\ b, a'\ b')$ parallel to the vertical plane of projection.

The vertical projection of the required cone is determined by drawing tangents through v', touching the ellipse forming the vertical contour of the ellipsoid in c' and d'. The plans of these points will be on the line $a\ b$; hence (c, c') (d, d') are two points on the required curve of contact. Similarly, the horizontal contour of the required cone is determined by drawing tangents through v touching the ellipse forming the horizontal contour of the ellipsoid in e and f; the vertical projections of these points will lie on the line $g'\ h'$ drawn parallel to the tangent at i'; hence two other points on the required curve of contact (e, e') (f, f') are determined.

Next, draw the axis of the cone $(v\ o, v'\ o')$, and determine the point (p, p') in which it cuts the upper surface of the ellipsoid, as in the last example. Also determine the two principal tangents through (p, p'); one $(p r, p' r')$ the tangent to the meridional generating curve through (p, p'), and the other $(q\ p, p' q')$ at right angles to the first. These tangents are constructed precisely as in the preceding problem. Next, through one of the points in the curve of contact

required, say (c, c'), draw a plane T R S parallel to the tangents at (p, p'); this is the plane of the required curve, and the points (j, j') in which this plane cuts the line $(v\,o,\ v'\,o')$ will be the centre of the required ellipse.

Through $(o,\ o')$ draw $(o\,k,\ o'\,k')$, $(o\,l,\ o'\,l')$ parallel to the tangents at $(p,\ p')$ as in the last example, and through $(j,\ j')$ draw $(j\,m,\ j'\,m')$, parallel to the tangent $(p\,q,\ p'\,q')$ and cutting the surface of revolution in $(m,\ m')$, on the circle of revolution through $(j,\ j')$ parallel to the equatorial circle; to determine the point $(n,\ n')$ in which the line $(j\,n,\ j'\,n')$ drawn parallel to $(p\,r,\ p'\,r')$ cuts the surface make $j\,n$ a fourth proportional to $o\,k$, $o\,l$ and $j\,m$; then $j\,m$, $j\,n$ will be the conjugate axes of the ellipse forming the horizontal projection of the required curve, and $j'\,m'$, $j'\,n'$ the conjugate axes of the vertical projection of the same; with aid of the circumscribing parallelograms and the other points already known, the required ellipses can be readily drawn.

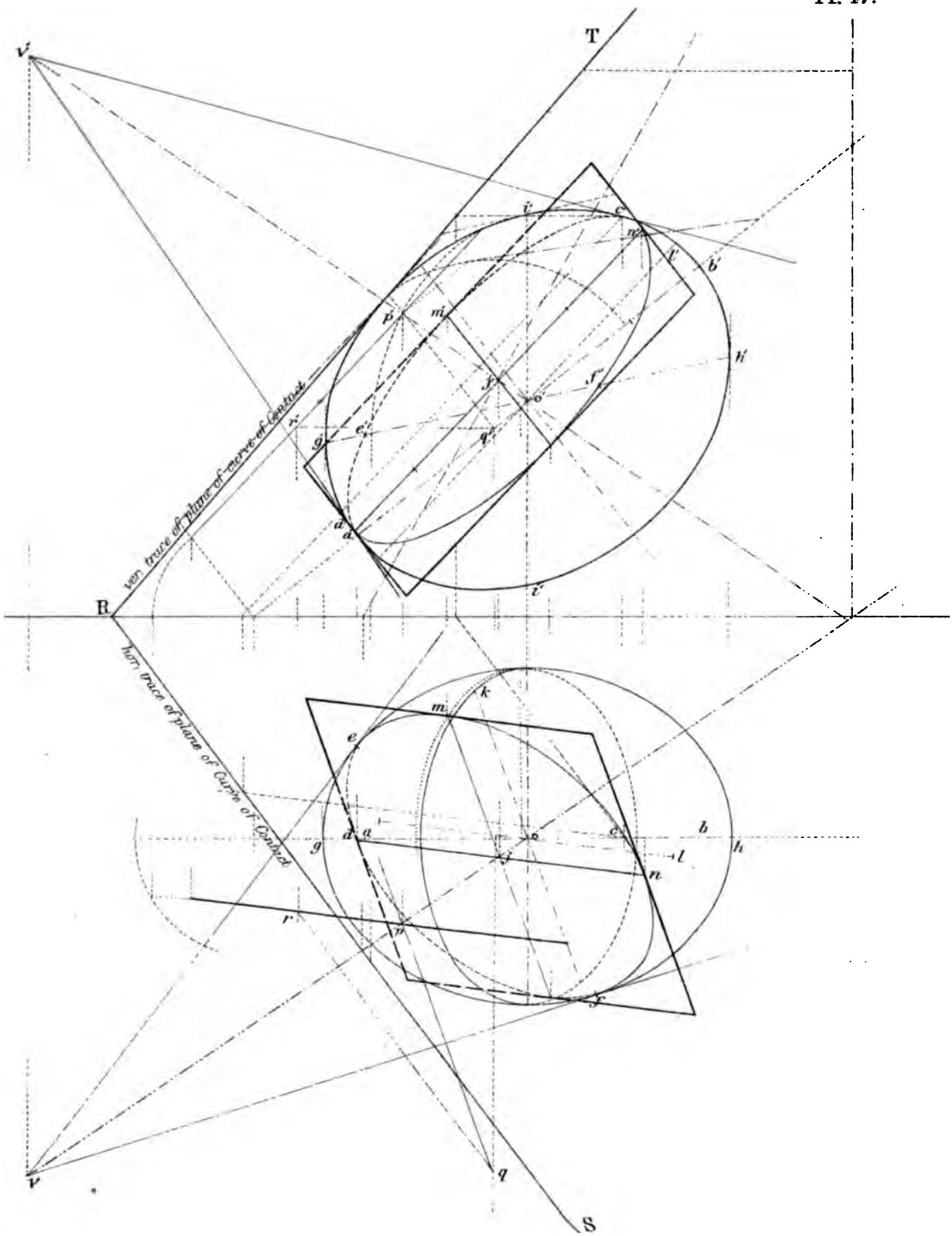
Note.—Before attempting this problem the student must have thoroughly mastered the constructions required in the preceding one.

86. To construct a cone to touch a given plane and have a given axis.

It is clear that the given axis must be inclined to the given plane; if it were parallel the touching surface would be a cylinder.

Let A B C (Plate XLVIII.) be the given plane, and $(v\,o,\ v'\,o')$ the given axis meeting the plane in $(v,\ v')$; this point is the vertex of the required cone.

Through any point $(o,\ o')$ in the given line draw a plane R S T perpendicular to $(v\,o,\ v'\,o')$. This will be the plane of the base of the cone. Find the line of intersection $(a\,t,\ a'\,t')$ of the given plane and plane of base; this line, since it lies in the given touching plane A B C, must be a tangent to the cone, and since it lies in the plane of the base of cone, it must be a tangent to this base. Through $(o,\ o')$ draw a perpendicular $(o\,p,\ o'\,p')$ to the given plane, cutting it in the point $p,\ p'$; then $(v\,p,\ v'\,p')$ will be the line of contact. Produce $(v\,p,\ v'\,p')$ to meet the line of intersection $(a\,t,\ a'\,t')$ in $(t,\ t')$, then $(t,\ t')$ will be a point in the required base of cone. Determine the real length of $o\,t$, $o'\,t'$ which will be the length of the major axis of the elliptical projections of the circular base. Having determined the minor axes of these projections, tangents through v and v' will complete the projections of the required cone.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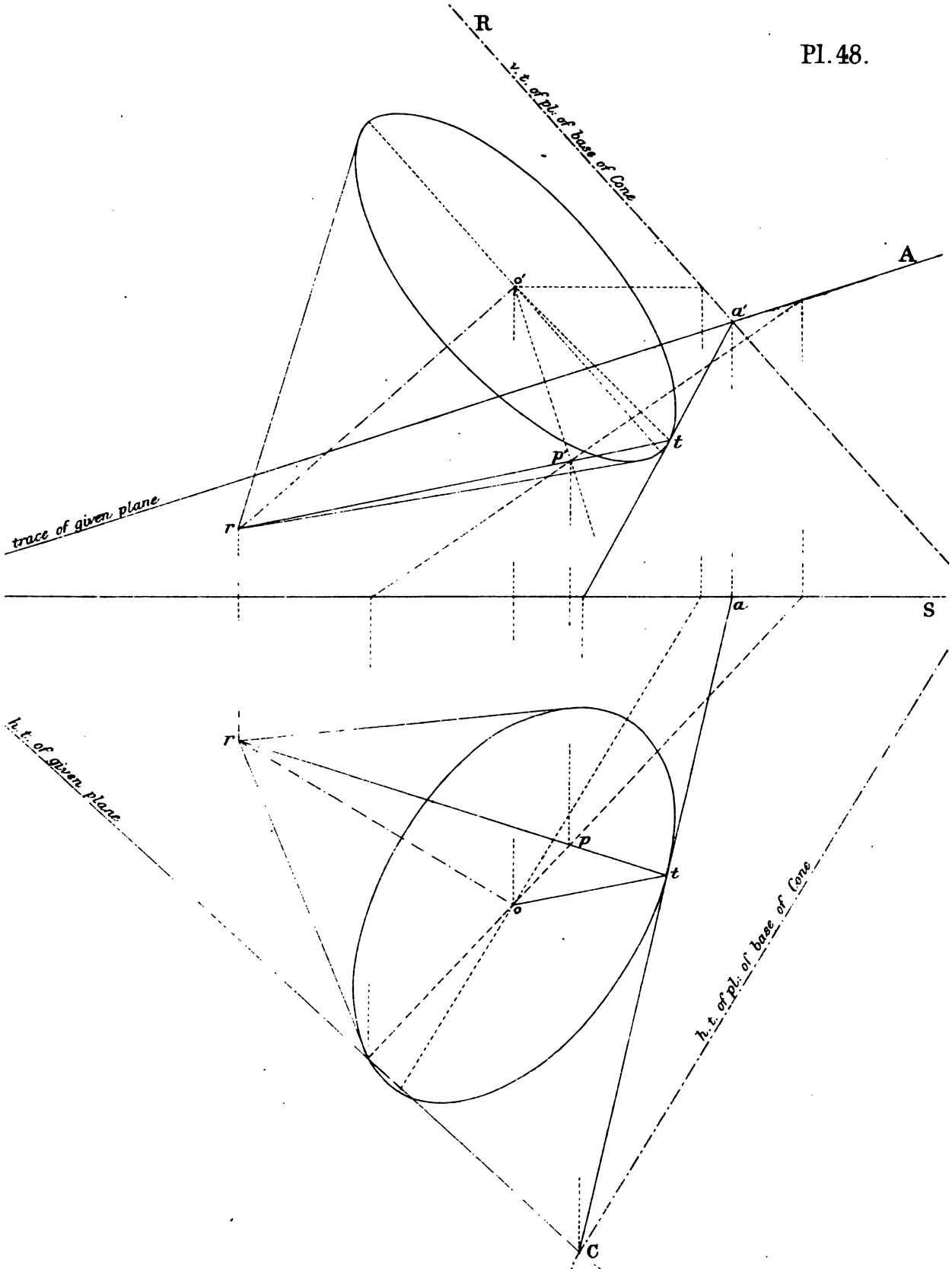
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C

PART II.

PERSPECTIVE OR RADIAL PROJECTION.

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1. Perspective is the science of representing on a plane surface, objects as they appear to the eye in any given position.

2. If straight lines be drawn from the boundaries of any proposed object to the eye of the observer, and they be cut by a plane, situated between the eye and the object, and lastly if the points of intersection be joined, an outline representation or perspective of the object will be drawn on the plane.

3. It will thus be seen that perspective projection differs from orthographic projection, simply in the fact that in the first the projectors or projecting lines radiate to a point, whereas in the second they are parallel.

4. As the intersection of a pyramid or cone is more difficult than that of a prism or cylinder, so the laws governing radial projection are more complex than those governing orthographic projection.

5. In constructing any perspective representation three things are generally given, the position of the vertex of the radiating lines or projectors, that of the plane of projection, which is nearly always vertical, and that of the object the representation of which is required.

6. Let L K in Plate I., represent the ground plane perpendicular to the plane of the paper, and let J I K be the plane of projection, perpendicular to the ground plane but inclined to the plane of the paper; V the vertex, and a β any given original line both out of the plane of the paper. Join V β, Va; these are the projectors of the given points, and the plane V a β is called the pro-

jecting plane of the line $a\ \beta$. Now since the projecting plane passes through V and $a\ \beta$ it will also pass through a line V p drawn through V parallel to $a\ \beta$; find p the point in which V p cuts the plane of projection J I K; p will evidently be a point in the perspective representation of the line $a\ \beta$.

7. The point p just found plays a most important part in perspective; for since the projecting planes of all lines parallel to $a\ \beta$ must also necessarily pass through V p, it follows that p is a point common to the projections of all lines parallel to $a\ \beta$, or the following important fact is proved, that the projections of any system of parallel lines all tend or appear to vanish in one point.

8. For this reason p is said to be the *vanishing point* of all lines parallel to $a\ \beta$.

9. Hence, to find the vanishing point of any line or system of parallel lines, draw through the vertex a line parallel to any one of them, and the point in which it meets the plane of projection, will be the vanishing point required.

10. It follows from the above that lines parallel to the plane of projection have no vanishing point, since a line drawn through the vertex parallel to the plane of projection can never meet that plane; therefore the projection of any line parallel to the plane of projection must be parallel to that line.

11. Draw V O perpendicular to the plane of projection and meeting it in O, then O is called the *seat of the vertex*, and is generally given in position in all problems, together with the perpendicular V O called the '*distance of the vertex*.'

12. This point O, the seat of the vertex, is also a very important point in other respects, for since V O is perpendicular to the plane of projection, O is the vanishing point of all lines parallel to V O or perpendicular to the plane of projection.

13. The point in which any original line meets the plane of projection is called the *trace* of that line; thus produce $\beta\ a$ to meet this plane of projection in t, then t is the trace of $a\ \beta$.

14. The perspective projection of any line, therefore, is the line joining its *trace* and *vanishing point*; thus tp is the perspective of the line $t\,a\,\beta$, and the farther off any point β be taken from t, the nearer will its projection approach the vanishing point p; thus p may be defined to be the projection of a point in $a\,\beta$ at an infinite distance from t. It now only remains to find the projection of any given portion $a\,\beta$ of the given line.

15. Through the line $t\,a\,\beta$ draw any plane E C D cutting the plane of projection in E C, and through V draw a plane parallel to the original plane E C D, cutting the plane of projection in G F. Since V p is parallel to $a\,\beta$, which lies in the original plane E C D, it is clear that V p must lie in the plane G V F; therefore, p must be a point in the line G F; and since $a\,\beta$ is any line in the plane E C D, it follows that the vanishing points of all lines lying in the plane E C D, or in any direction parallel to that plane, all lie on the line G F. For this reason G F is called the *vanishing line* of the plane E C D, and of all planes parallel to E C D. And the line E C is called the *trace* of the plane E C D.

16. A plane is said to be given when its trace and vanishing line are known; a series of parallel planes can have but one vanishing line, but must have a distinct trace for each plane.

17. Planes which are parallel to the plane of projection, can have neither vanishing lines nor traces, and are given by their perpendicular distance from the vertex.

18. Since O is the vanishing point of all lines perpendicular to the plane of projection, it follows that the vanishing lines of all planes perpendicular to the plane of projection must pass through O.

19. Since the plane of projection is generally assumed to be vertical, it follows from the last section that the vanishing line of all horizontal planes is a horizontal line drawn through O; also, that the vanishing lines of all vertical planes perpendicular to the plane of projection, is a vertical line drawn through O.

20. Through V draw V o perpendicular to G F, then o is said to be the centre of the vanishing line G F, being the nearest point in that line to the vertex; join o O, then the plane V o O is perpendicular to the plane of projection

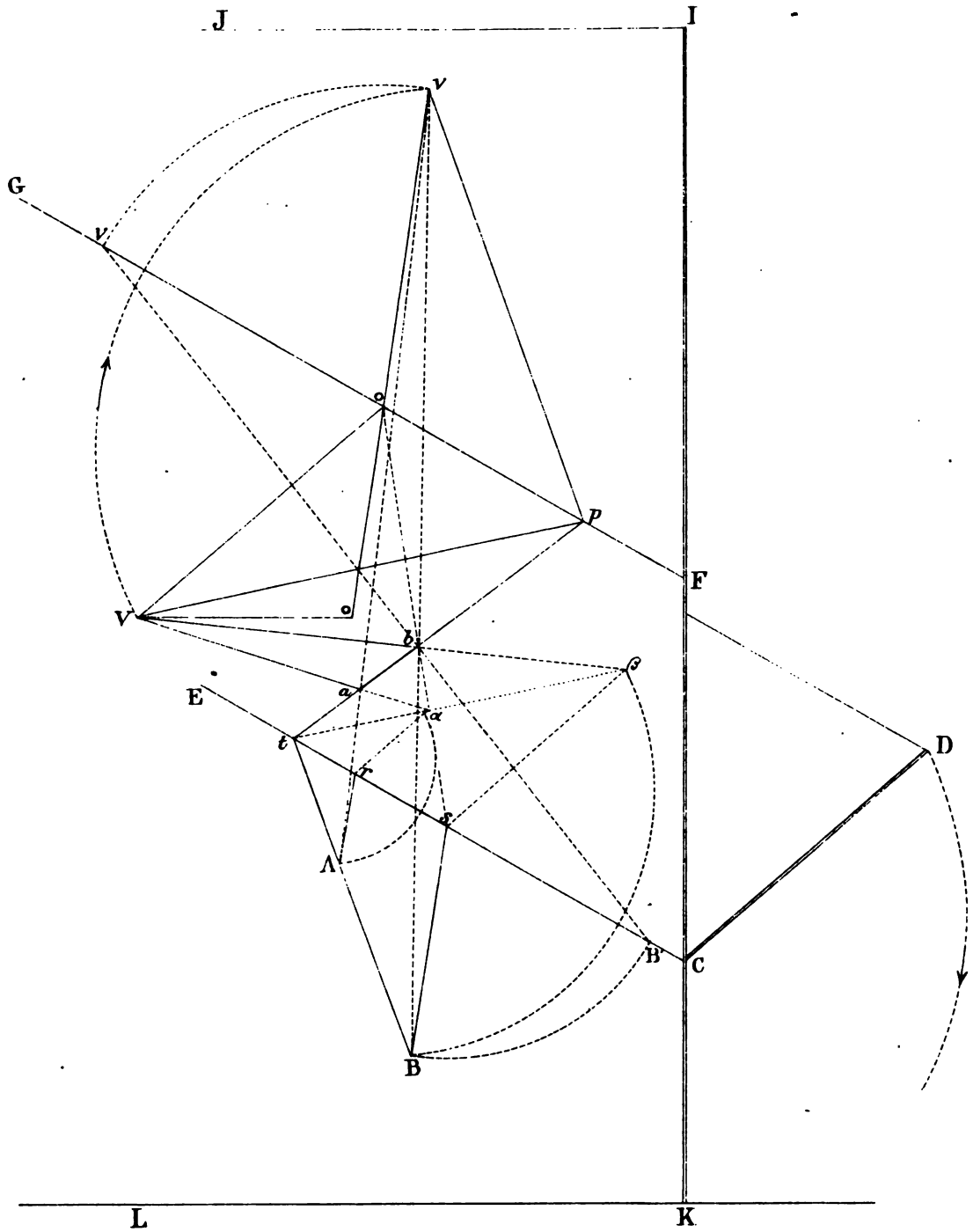
and the original plane E C D, or vanishing plane V G F, and the angle V o O is the angle of inclination of the original plane E C D to the plane of projection, and the line o O drawn through O perpendicular to G F is the vanishing line of all planes perpendicular to the plane of projection and the original plane.

21. In the plane E C D draw $a r$, βs, perpendicular to the trace of the plane E C and join $r o$, $s o$. Since V o is parallel to $r a$ and $s \beta$, therefore o is the vanishing point of these lines; and r, s, being their respective traces, $r o$ and $s o$ are respectively the projections of $r a$ and $s \beta$.

(13). Now, a, β, are points common to the two sets of original lines $t a \beta$ and $r a$, and $t a \beta$ and $s \beta$; and, therefore, the points a, b, common to the projections of these lines, will be the perspective projections of a and β. It would have been simpler to have drawn lines from a and β perpendicular to the plane of projection, as in that case O would be the vanishing point. It only remains to see how these operations can all be represented on one plane, namely, the plane of projection or of the paper. To effect this, suppose the planes V G F, E C D, to revolve about the lines G F, E C, until they coincide with the plane of projection. Then if O o drawn perpendicular to G F be produced to v so that $o v$ is equal to O V, v will be the new position of the vertex, and in a similar manner the points A and B are found; and since during the revolution the planes V G F, E C D, are supposed to remain parallel, therefore $v p$ will still be parallel to t A B, and the perspective $a b$ of $a \beta$ is found by joining $t p$ and finding the points in which $o s$ and $o r$ intersect it. There are many other methods of determining the perspective points, depending upon the following useful and simple principle. Since the triangles V $o b$ and $\beta s b$ are in the same plane and similar, therefore V o is to βs as $o b$ is to $b s$, and this ratio manifestly holds good so long as o V and $s \beta$ are parallel and in the same plane; hence in their new position $o v$ is to s B as $o b$ is to $b s$, or the points a and b may be found by joining A v, B v, and finding where they intersect $t p$. By the same principle, the point b may be found by making $o v'$ along G F equal to $o v$, and s B′ in the opposite direction along E C equal to s B and joining v' B, it will also cut $t p$ in the same point b.

22. We will next apply the principles enunciated in the preceding articles to the solution of the following problem:—

Having given the position of the vertex O and its distance O V, also the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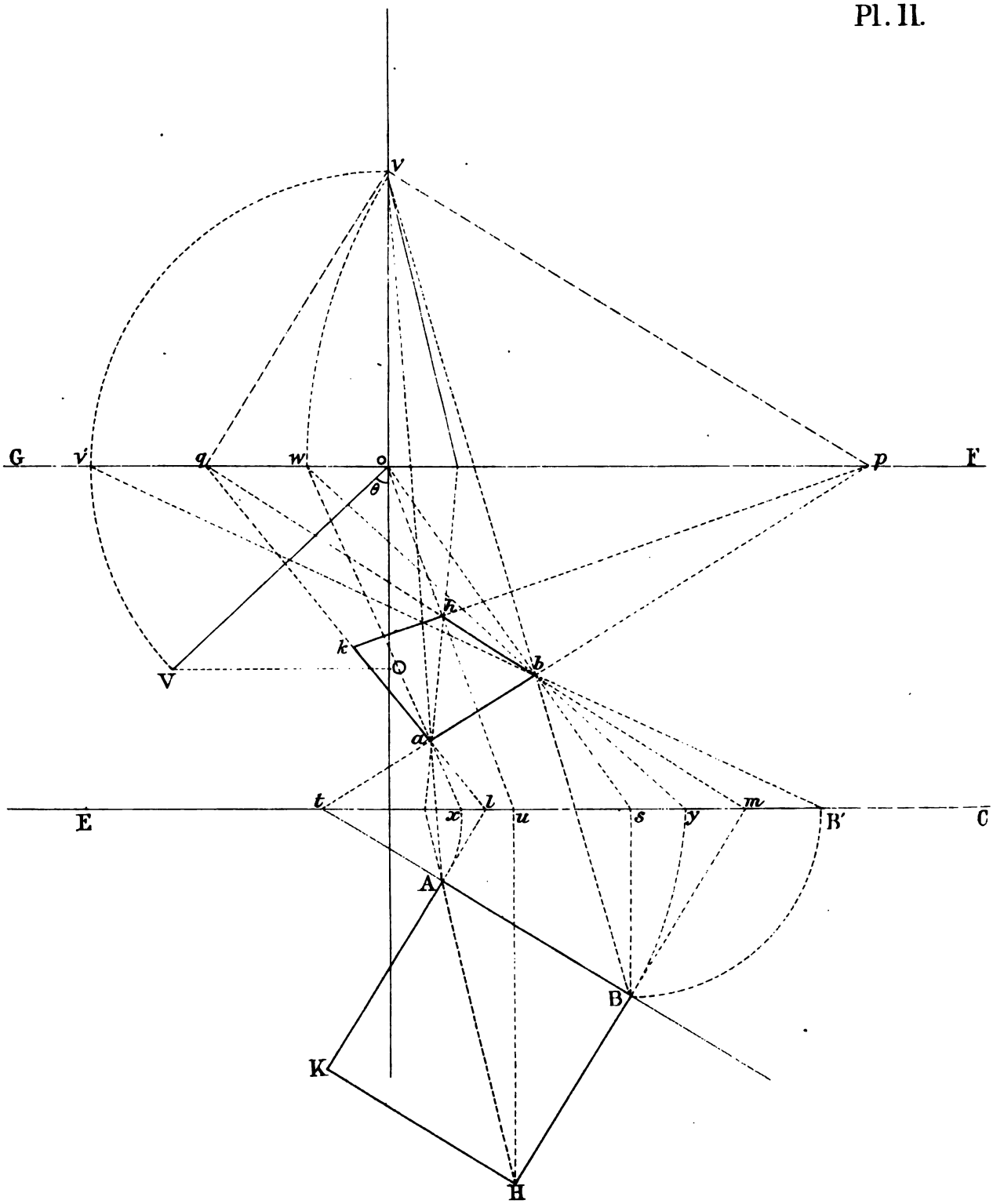
Pl. 11.



trace E C of a plane inclined to the plane of projection at a given ang'e $\theta°$; and the position of a square A B C D in that plane; to determine its perspective projection.

(Plate II.) Draw O o perpendicular to E C, and O V perpendicular to O o, make the angle V o O equal to $\theta°$, the given inclination of the plane of the square. Through o draw G o F parallel to E C, it will be the vanishing line of all planes inclined to the plane of projection at the angle θ. Make $o\,v$ in O o produced equal to o V; produce B A to meet E C in t, the trace of the line A B, and draw $v\,p$ parallel to A B, cutting G F in p, the vanishing point of all lines parallel to A B; join $t\,p$; then tp is the perspective of t A B, and the points a, b, in which this line is cut by v A, v B, will be the perspective projections of A and B respectively. These points, as before, may also be found several other ways; for instance, draw B s perpendicular to E F and join $s\,o$; then $s\,o$ being the perspective projection of s B, it must also pass through the perspective projection of B or b. Again, make s B′ equal to s B along E F, and $o\,v'$ in the opposite direction equal to $o\,v$ along G F and join v' B′, this line will also pass through b. Through v draw $v\,q$ parallel to A K or perpendicular to $v\,p$, cutting G F in q, then q is the vanishing point of all lines parallel to A K; join $q\,a$, $q\,b$, the directions of the perspective projections of the sides A K, B H; determine the projection of the point H by any one of the methods shown above, viz.: by drawing H u perpendicular to E C and joining $u\,o$; the point h in which $u\,o$ cuts $b\,q$ will be the projection required. Lastly, join $p\,h$ and produce it to cut $a\,q$ in k, and the projection $a\,b\,h\,k$ of the given square will be completed.

23. It is not necessary that the original square A B K H should be drawn in order to find its perspective; it will be sufficient if we know the trace t of A B, the angle which A B makes with the trace of the plane E C F G,* and the distance of A and B from t. For having drawn $v\,p$ inclined to G F at the given angle, join $t\,p$. Now, if $t\,x$ be made equal to t A, and $t\,y$ equal to t B along E C, and $p\,w$ equal to the radial $p\,v$ set out along G F in the reverse direction; by joining $w\,x$, $w\,y$ the points a, b, in which they cut $t\,p$, are readily obtained. The principle upon which the truth of this method depends, has been fully explained in Art. 21. By this means also a line may be divided in any determinate manner. Now bisect the angle $q\,v\,p$ by the line $v\,z$; $v\,z$ is parallel to the diagonal A H, and therefore z is the vanishing point of this line; join $z\,a$, $q\,b$, and their point of

* For the sake of brevity we shall in future denote a plane by its trace and vanishing line, thus the plane E C F G is an abbreviation for the plane which has E C for its trace and F G for its vanishing line.

intersection h will give a third corner of the square; lastly, draw $p\,h$ and produce it to meet $q\,a$ in k, and the square will be completely projected. By similar constructions, the projection of any plane figure, howsoever inclined to the plane of projection, may be determined.

24. We may now extend the preceding problem to that of finding the projection of a cube, having given the plane of one face E C, G F and, the trace t and inclination of an edge in that face to the trace of the plane, together with the actual distances of the extremities of this edge from its trace.

(Plate III.) Having found the projection $a\,b\,h\,k$ of the given face by any one of the preceding methods, it only remains to draw through these points lines representing the projections of perpendiculars to the given plane, and making them the requisite height. By referring back to Art. 19 and Plate I., it is there shown that the line O o is the vanishing line of all planes perpendicular to the plane of projection and the original plane (E C, F G).

Hence it will be seen that the vanishing point of lines perpendicular to the plane (E C, F G) must lie on the line O o. The triangle o V O (in Plate III.) is supposed turned down into the plane of projection about o O, and since V o represents the principal radial of the plane (E C, F G,) a line drawn through V perpendicular to V o to meet o O in x, will be the radial of lines perpendicular to the plane (E C, F G,) and the point x their vanishing point. Join $x\,a$, $x\,b$, $x\,h$, and $x\,k$ and produce them; these will be the directions of the perpendiculars. Since x and q are the respective vanishing points of the lines $a\,a$, $a\,k$, the line $x\,q$ will be the vanishing line of the face $a\,k\,a\,\delta$, and its parallel face $b\,\beta\,\gamma\,h$; similarly, the line $x\,p$ is the vanishing line of the planes of the parallel faces $a\,a\,b\,\beta$ and $k\,\delta\,h\,\gamma$. To find the point a or the projected length of the edge $a\,a$, draw O o' perpendicular to $x\,p$ as the vanishing line of planes perpendicular to the planes $x\,p$ and the plane of projection; draw O V′ perpendicular to O o' and equal to the distance of the vertex, and join V′ o', then the angle O o' V′ will be the angle at which the planes $x\,p$ are inclined to the plane of projection. Lastly, make $o'\,v'$ in O o' produced equal to o' V′, and join $v'\,p$, $v'\,x$ the radials of the sides $b\,\beta$, $a\,a$. Bisect the angle $x\,v'\,p$ by the line $v'\,w$, meeting $x\,p$ in w; then $v'\,w$ is the radial of the diagonal $b\,a$ and w its vanishing point; join $w\,b$ and produce it to meet $a\,a$ in a, the required extremity of the edge $a\,a$. This construction is generally required for every plane face of a solid; in the present case, an easier construction would have sufficed, viz., by drawing a semicircle on $x\,p$ as diameter and the point in which O o' produced cuts the circle will give the point v'; and

generally where the vanishing points of lines are given inclined to each other at a given angle, all that is necessary is to construct a segment of a circle containing the given angle on the line joining the vanishing points and the point of intersection of this segment with the perpendicular from O on to the vanishing line, will give the constructed position of the vertex. To continue with the completion of the problem we are about, it only remains to join $a\,p$, $a\,q$, cutting $b\,\beta$, $k\,\delta$ in β and δ, and lastly, by joining $\beta\,q$, $\delta\,p$, we shall obtain the point γ, the extremity of $h\,\gamma$ and the required projection of the cube will be completed.

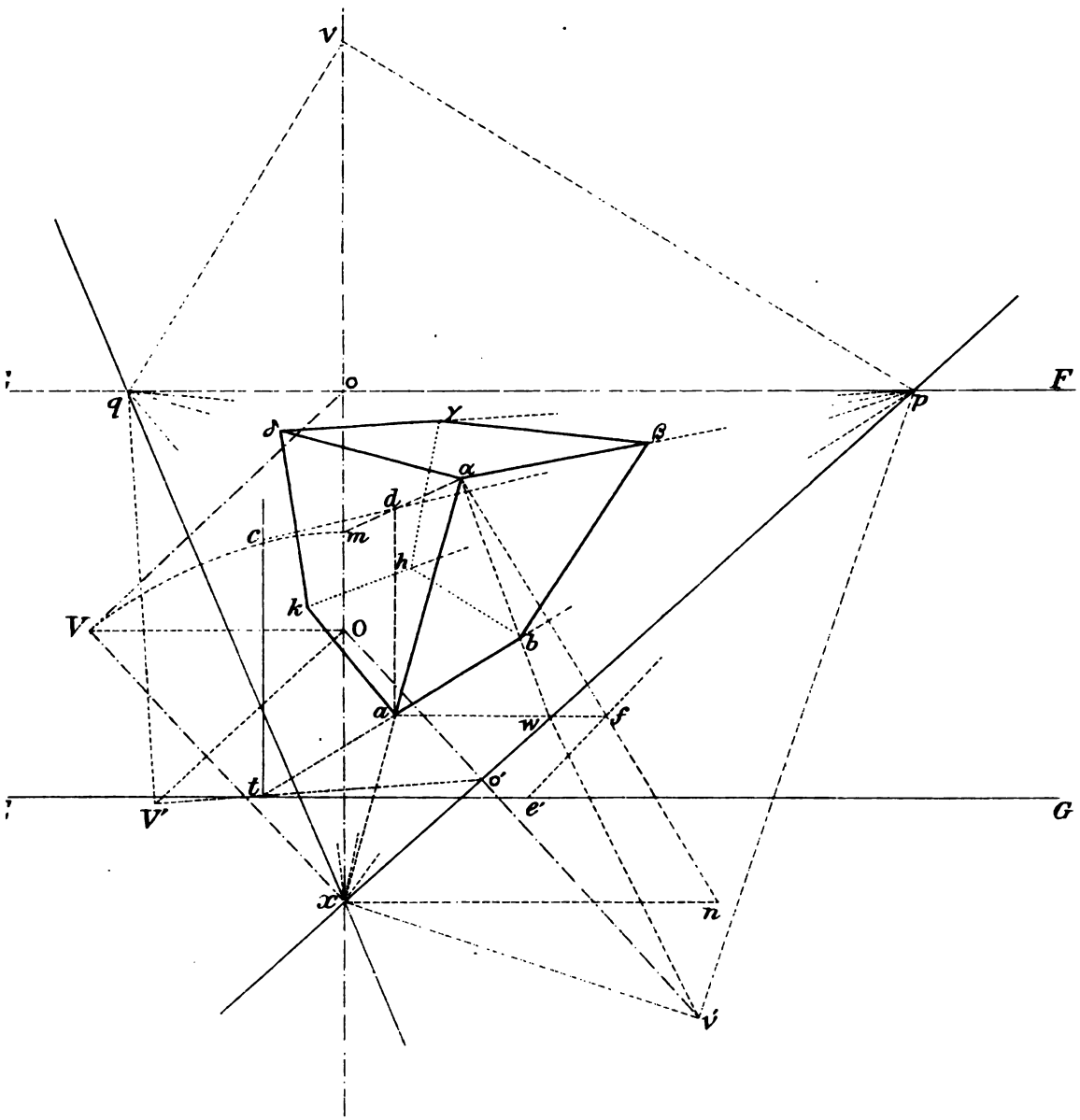
The point a might have been found in several other ways of which we give two. Through a draw a line $a\,d$ parallel to O o or the projection of an original line parallel to the plane of projection; through t draw $t\,c$ parallel to $a\,d$ and equal to the real length of the edge of the cube. Join $c\,p$ cutting $a\,d$ in d. Then $a\,d$ is obviously the perspective of a line equal in length to $t\,c$. Next construct the radial x V of the line $a\,a$, along x O o in the point m; join $m\,d$ and produce it to meet $x\,a$ in a; then $a\,d$ and $a\,a$ will be the projections of lines of equal length. The reason of this construction is the same as that given in Art. 23 for finding $t\,a\,b$. A second similar method is the following: draw $t\,e$ along E C equal to $t\,c$ or the given edge, and draw $a\,f$ parallel to E C. Join $e\,p$, then $a\,f$ will be the perspective projection of a line equal to $t\,e$. Draw $x\,n$ parallel to $a\,f$ and equal to x V, join $n\,f$ and produce it to meet $a\,a$ in a.

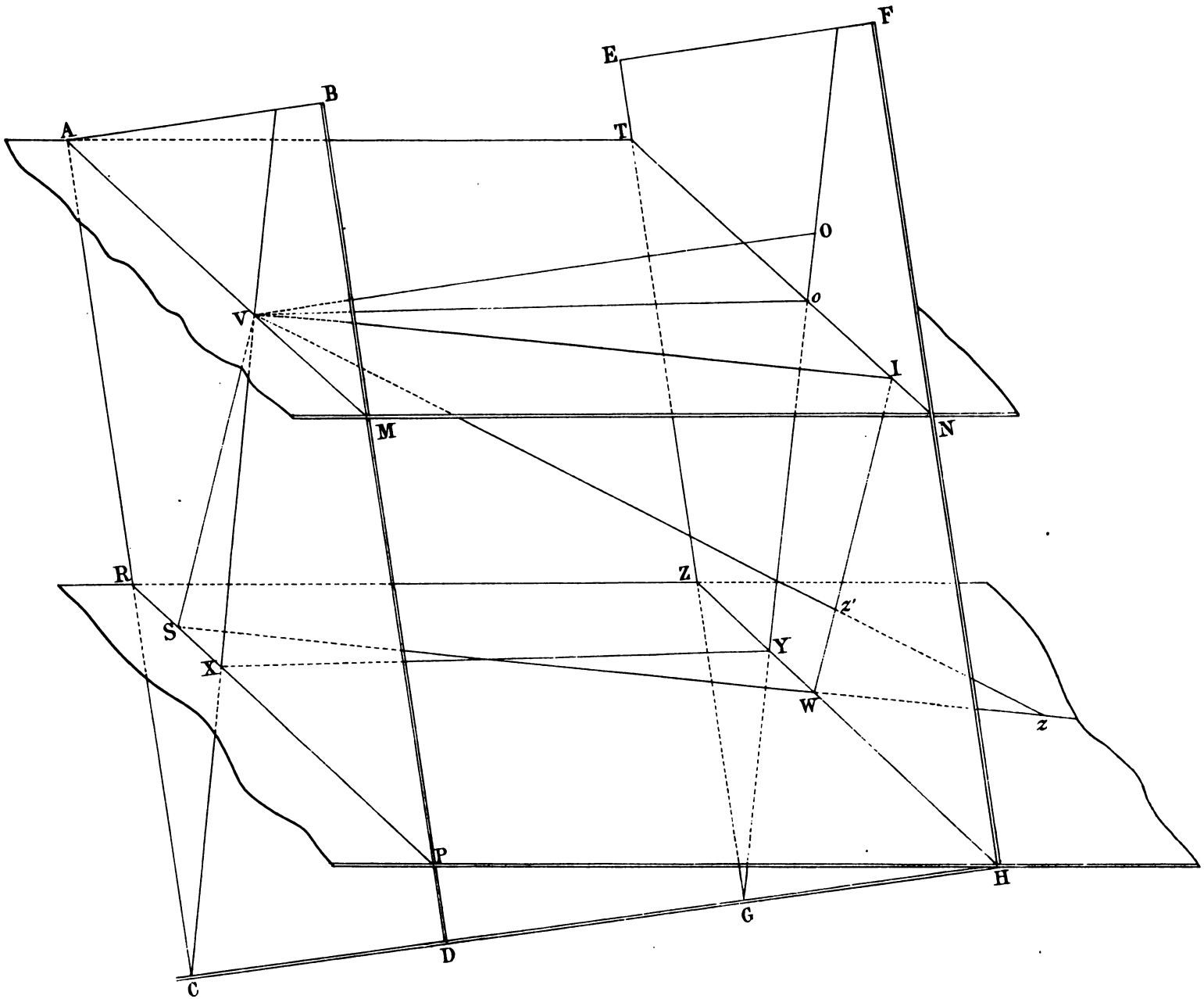
25. Before proceeding farther, there are a few other auxiliary planes, lines, and points necessary for the construction of more difficult problems. (Plate IV.) Let E F G H be the plane of projection, perpendicular to the horizontal plane C H, here supposed to be viewed orthographically and perpendicular to the plane of the paper. Let V be the position of the vertex of the pyramid of projectors. V O drawn perpendicular to E F G H is the distance of the vertex and O the centre of the plane of projection. Through V draw a plane A B C D parallel to the plane of projection E F G H, this is called the *directing plane*. If an original plane be not parallel to the plane of projection and directing plane, it must cut them if produced. Let R P H Z be an original plane cutting the directing plane in R P and the plane of projection in Z H; R P is called the *directing line*, and Z H the *intersecting line* or *trace* of the original plane R P Z H. Through V draw the plane A M T N parallel to the original plane R P Z N and cutting the directing plane in A M and the plane of projection in T N. Then, as we have seen, T N is the vanishing line of R P Z H and all parallel planes; A M is called the *parallel of the vertex* of the plane R P Z H. Since A M, T N, R P, and R H are the intersections of parallel planes with parallel planes,

they are all parallel to one another. The plane drawn through V O, the distance of the vertex, perpendicular to an original plane, is termed the principal plane of the original plane. This principal plane to R P Z H is the plane V *o* Y X and is perpendicular to the plane of projection and the directing plane, because it contains V O, which is perpendicular to each of them; it is also by construction perpendicular to the original plane and its vanishing plane; it will therefore cut the intersections of these planes at right angles, or be perpendicular to the vanishing line T N, the parallel of the vertex A M, the trace of the original plane Z H and the directing line R P. Hence V *o* Y X is a parallelogram. The intersection V X of the principal plane V *o* Y X with the directing plane A B C D, is called the *director of the vertex*; this line represents the height of the vertex when the original plane is the ground plane or horizontal. The intersection O *o* Y of the principal plane with the plane of projection E F G H, is called the *perpendicular line of the original plane*, and the point *o* in which it cuts the vanishing line T N is called the *centre* of that vanishing line. The intersection V *o* of the principal plane with the vanishing plane, is called the *radial of the original plane*, or the *distance of the vanishing line*; and lastly, the intersection X Y of the principal plane with the original plane, is called the *line of station* of the original plane. Let S W be an original line lying in the original plane R P H Z, and not parallel to the plane of projection; it will, therefore, if produced meet the plane of projection in some point W, and the directing plane in some point S; the first is called the *trace* of the original line and the other, S, the *directing point* of the original line. Find the vanishing point I of the line S W, by drawing V I parallel to S W meeting the vanishing line of the plane containing S W in I. V I is *the radial* of the original line. The plane V I S W is called the *projecting plane* of the original line; W I its intersection with the plane of projection is obviously the perspective projection of S W, and and its intersection V S with the directing plane is called the *director* of the original line. Then V I W S is a parallelogram.

26. From the preceding definitions, it follows that all original lines which have their directing points in the same director must have their perspective projections, parallel, because all the projecting planes must pass through the same director, and hence will cut the plane of projection in lines parallel to this director; also, all lines in an original plane must have their vanishing points, traces, and directing points respectively in the vanishing line, intersecting line or trace, and directing line of the original plane.

Pl. III.

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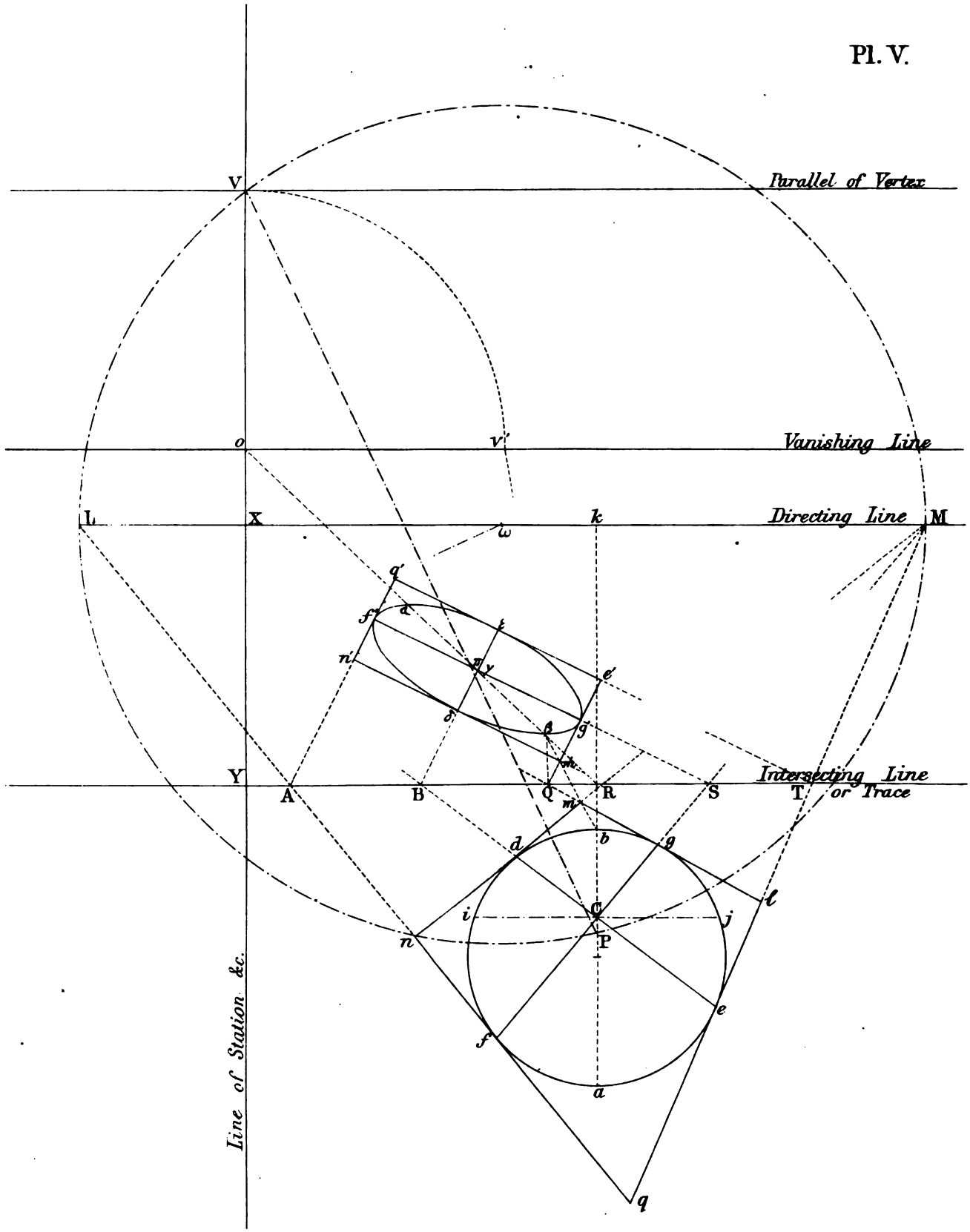
Pl. V.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27. On referring to the plate, it will be seen that the vanishing plane, original plane, plane of projection, and directing plane, may be conceived to be hinged together at their lines of intersection, and these being all parallel, the planes may be turned in any direction whatever, without affecting the position of any perspective projection of any figure in the revolving plane, and it is the truth of this principle which enables us to conceive all these several planes as revolved until they all lie in one plane, the plane of the paper on which the several constructions are drawn and the projection determined. It will also be observed that when the several planes are so adjusted, the four several hinge lines will still be parallel to each other and their several distances preserved, and these distances are given by the parallelogram V *o* Y X.

28. (Plate V.) In this plate we have supposed the four planes (shown in Plate IV.), viz. the vanishing plane, original plane, plane of projection, and directing plane, to be revolved about their lines of intersection until they coincide with the plane of projection or of the paper; the four lines of intersection, the parallel of the vertex, the vanishing line, directing line, and intersecting line, will all remain parallel, and V X will be equal to *o* Y and V *o* to X Y, and the principal plane V *o* Y X being perpendicular to the plane of projection which is here the plane of the paper, will be represented by a straight line perpendicular to the four hinge lines. Having arranged the planes of construction in the manner above described, let it be proposed to find the projection of a circle *f d g e*, situate in the original plane Y T, *o v'*. Since the plane of projection lies between the vertex and the original circle, the perspective projection must be an ellipse because the sections of a cone produced by planes cutting between the vertex and base are either ellipses or circles. If the original circle touched the directing line, the plane of projection would necessarily be parallel to a side of the cone and hence the section of the cone, which is the perspective of the circle, would be a parabola; and lastly, if the circle cut the directing line, the perspective projection is an hyperbola.

In the present case, the perspective might be determined by finding the perspective projections of several points in the circle and tracing the curve through them; but the more accurate and scientific method is to determine the major and minor axis of the required ellipse, whence the curve may be readily drawn by a new method which will be described in the next article.

Before proceeding further it is necessary to establish the following theorem: that (Fig. 1, Plate VI.) if four harmonical lines *v f*, *v c*, *v g*, *v m* be drawn, then

a line $f'\gamma g'$ drawn parallel to any one of them, $v m$, will be bisected by the other three, or $f'\gamma = \gamma g'$. To prove this, through γ draw the line $a b d$ parallel to the original line $f c g m$, then $a d$ will be harmonically divided in a, γ, b and d;

$$\therefore a d : a\gamma :: b d : b\gamma. \quad . \quad . \quad (1)$$

But from the similar triangles $\mathrm{V}\, d\, a$, $f'\gamma a$,

$$\mathrm{V}\, d : f'\gamma :: a d : a\gamma,$$

and from the similar triangles $\mathrm{V}\, d\, b$, $b\gamma g'$,

$$\mathrm{V}\, d : \gamma g' :: b d : b\gamma;$$

$$\therefore \text{from (1) } \mathrm{V}\, d : f'\gamma :: \mathrm{V}\, d : \gamma g';$$

$$\text{therefore } f'\gamma = \gamma g'.$$

Now we may suppose the line $f g$ to be an original line, and m its directing point; also that $f'\gamma g'$, drawn parallel to the director $v m$, is the perspective projection of $f c g$, and it follows that if one of the points of harmonical division of an original line be its directing point, that the perspective of the line will be bisected by the perspectives of the other three points.

To return to the problem:—through the centre of the given circle draw $a b k$ perpendicular to the directing line L M, and cutting it in k; draw the chord of tangents from k, $i j$ cutting $a b$ in C; then, if any line be drawn through k, such as L M, and from any point in it, as M, tangents be drawn to the circle M d, M e, the chord $d e$ must pass through C; and if the chord $e d$ be produced to meet the same line in L and a semicircle be described on L M, this semicircle will always cut the line $k b a$ in a point P, such that k P is equal to the tangent $k j$.

Join V P, and bisect it by a perpendicular cutting the directing line in ω; with ω as centre and ω V or ω P as radius, describe a circle cutting the directing line in L and M; from L and M draw the tangents Lf, L g, M d, M e, then the chords $d e$, $f g$ will cut each other in the point C. Find the perspective of the line f C g—viz., $f'\gamma g'$; then, since f C g m is harmonically divided in these points, and one of them, m, is its directing point, therefore f' g' is bisected in γ. In like manner the perspective of d C e—viz., $\delta\gamma\varepsilon$, is bisected in γ; therefore γ is the centre of the ellipse required, and because the directors of the lines $d e$, $f g$—viz. V L, V M, are at right angles to each other, being in the semicircle L V M; therefore the perspectives $f'g'$, $\delta\varepsilon$, which are parallel to these directors, must be also perpendicular to each other, and they are therefore the major and minor axes of the ellipse; whence the curve may be readily drawn. A good approximate method of drawing an ellipse by means of circular arcs will next be given.

Fig. 1.

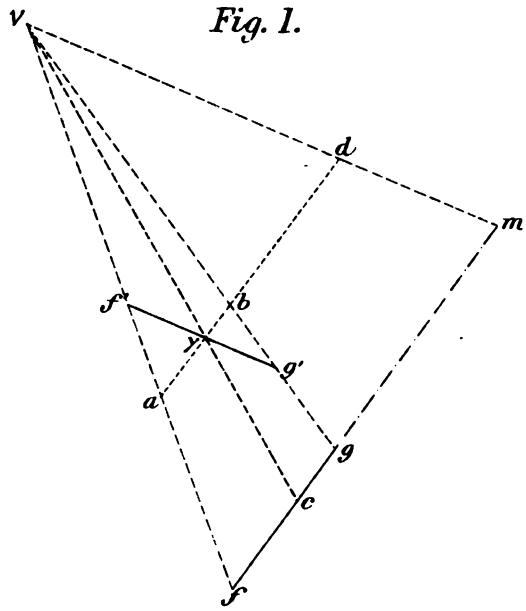
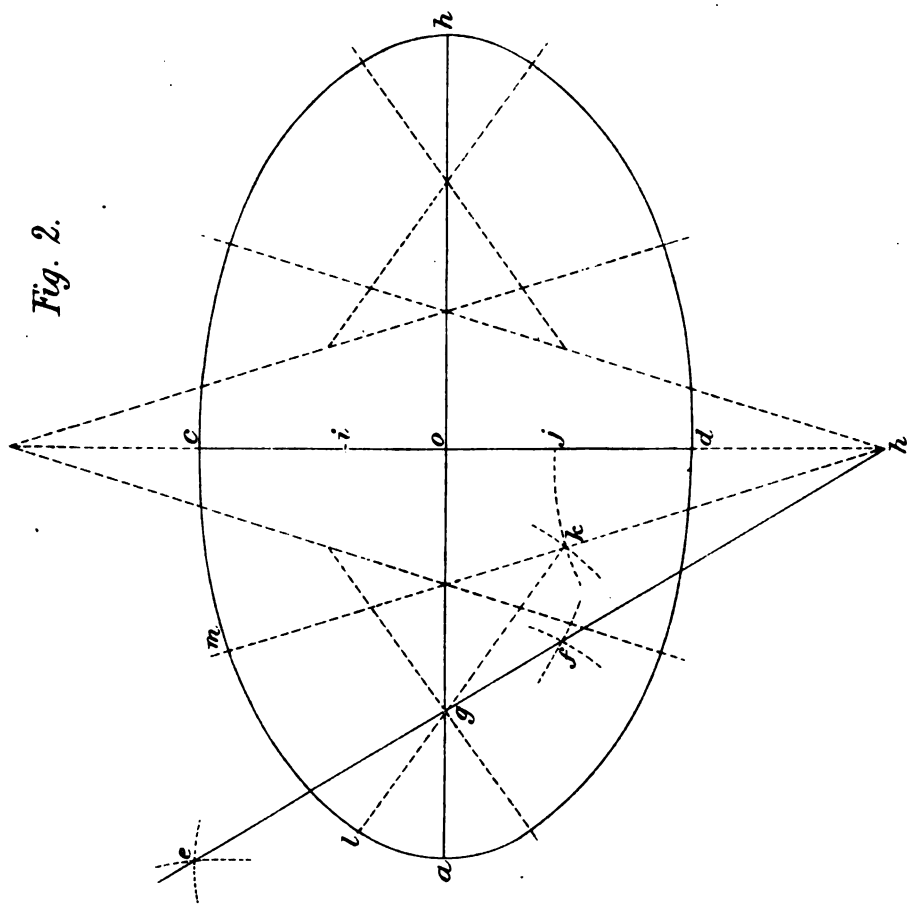
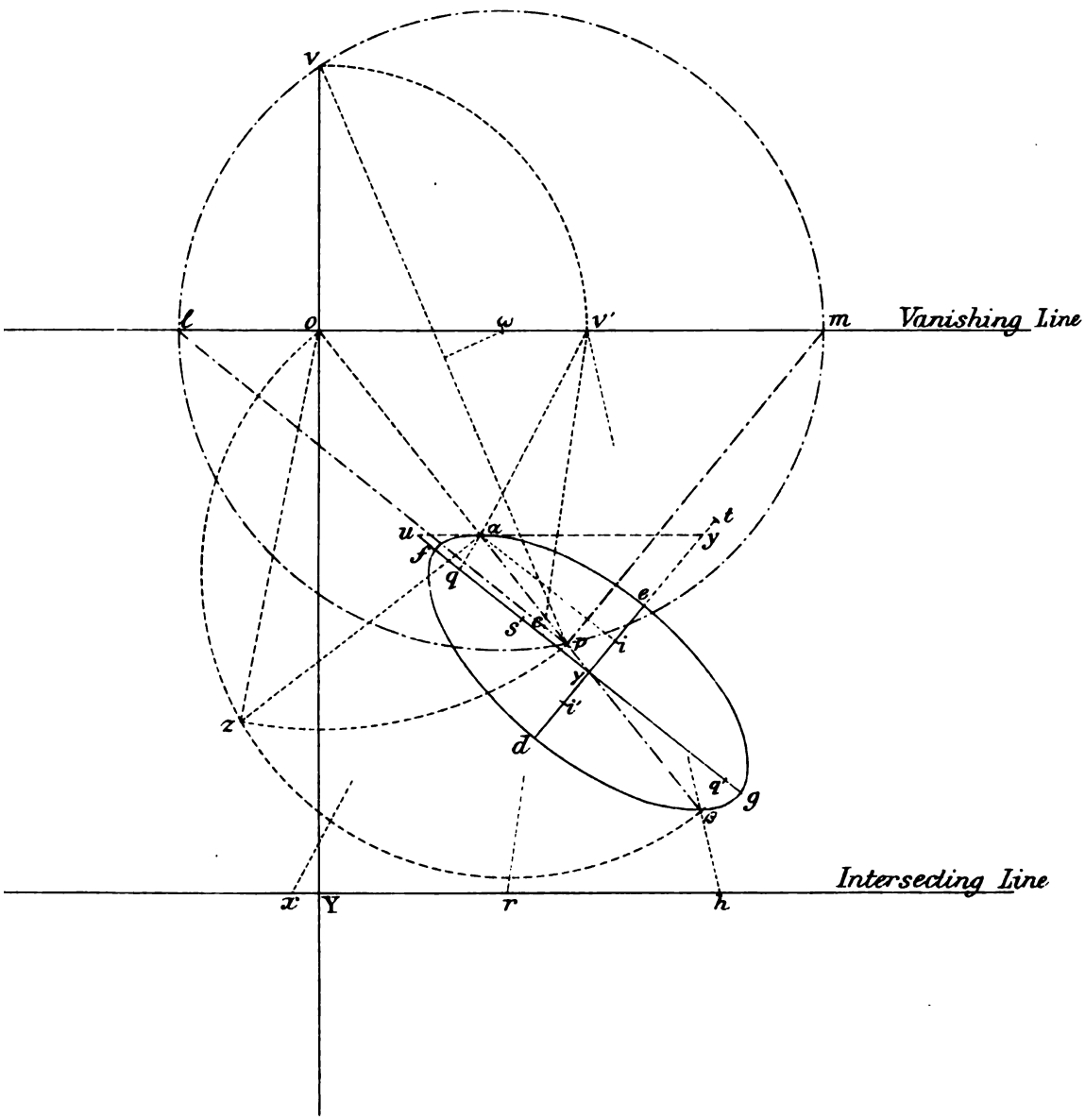


Fig. 2.

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29. (Plate VI.) To draw an ellipse approximately with three centres, having given the major and minor axes.

Let $a\,o\,b$, $c\,o\,d$ be the two axes. With a as centre and $o\,c$ as radius, and with c as centre and $a\,o$ as radius, describe arcs intersecting in e and f; join $e\,f$, and produce it to cut the major axis in g, and the minor in h; make $c\,i$ equal to $g\,a$, and divide $h\,i$ in j so that $h\,j$ is to $j\,i$ as $o\,a$ is to $o\,c$. With g as centre and $i\,j$ as radius, and h as centre and $h\,j$ as radius, describe arcs cutting in k; join $k\,g$ and produce, also $h\,k$ and produce. Lastly, with g as centre and $g\,a$ radius, describe the arc $a\,l$, cutting $k\,g$ in l; with k as centre and $k\,l$ radius, describe the arc $l\,m$, cutting $h\,k$ in m; and with h as centre and $h\,m$ as radius, describe the arc $m\,c$. Similarly, for the other three quadrants. The curve thus produced will not differ very materially from the true ellipse.

30. The perspective of that diameter of a circle which is perpendicular to the directing line of its plane being given, thence to determine the axes of the ellipse. (Plate VII.)

Let o be the centre of the vanishing line of the plane of circle, and therefore the vanishing point of the given diameter $\beta\,a$. Step out the distance of the vertex $o\,v'$ along the vanishing line; join $v'\,a$, $v'\,\beta$, and produce to meet the intersecting line in x and h; bisect $x\,h$ in r, and join $v'\,r$, cutting $\beta\,a$ in c; then c is the perspective of the centre of the original circle. Bisect $\beta\,a$ in γ the centre of the ellipse. Find p in $a\,\beta$, a mean proportional between $o\,a$ and $o\,\beta$, and join $v\,p$. Bisect $v\,p$ by a perpendicular cutting the vanishing line in ω. With ω as centre and $\omega\,v$ or $\omega\,p$ as radius, describe a circle cutting the vanishing line in the points l and m, and join $l\,p$, $m\,p$; lastly, through γ, the centre of the ellipse, draw $f\,\gamma\,g$ parallel to $l\,p$, and $e\,\gamma\,d$ parallel to $m\,p$, and $f\,g$, $e\,d$ will be the directions of the indefinite axes.

On referring to the original figure in Plate V. it will be seen that the situation of L and M with respect to P in the original plane, is such that lines drawn from L and M to P, are not only perpendicular but have perpendicular perspectives. Now, because of the circle $v\,l\,m\,p$ (in Plate VII.), whose diameter is $l\,m$, the angles $l\,v\,m$, $l\,p\,m$ are right, and therefore $l\,p$ and $m\,p$ are perpendicular, and consequently represent L P and M P in the original plane, there being no two other lines which can pass through p with these conditions, seeing that ω is the only point in $l\,m$, from which, as a centre, a circle can be drawn to pass through v and p. And because of the directing points L and M (in Plate V.), the images of L C, M C which pass through C (the original of the

centre of the ellipse) are parallel to the images of L P, M P; therefore the indefinite lines (in Plate VII.) $f\gamma g$, $e\gamma d$ drawn through γ parallel to lp, mp, represent L C and M C in the original plane, and are therefore the indefinite axes required.

The length of the axes may now be readily determined thus. Through a draw $u\,a\,v$, a tangent to the ellipse at a, and therefore parallel to the vanishing line $l\,m$. This must be so, because the original of the tangent is perpendicular to the given diameter of the circle, which is known to be perpendicular to the directing line of its plane. Produce $f\,g$ and $d\,e$ to cut the tangent in u and y; draw $a\,q$ perpendicular to $f\,g$, and make γf a mean proportional between γq and γu, then γf will be half the major axis; in like manner make γe a mean proportional between γi and $\gamma\ y$, and γe will be the semi-minor axis, whence the whole ellipse may be readily constructed without having recourse to the original figure.

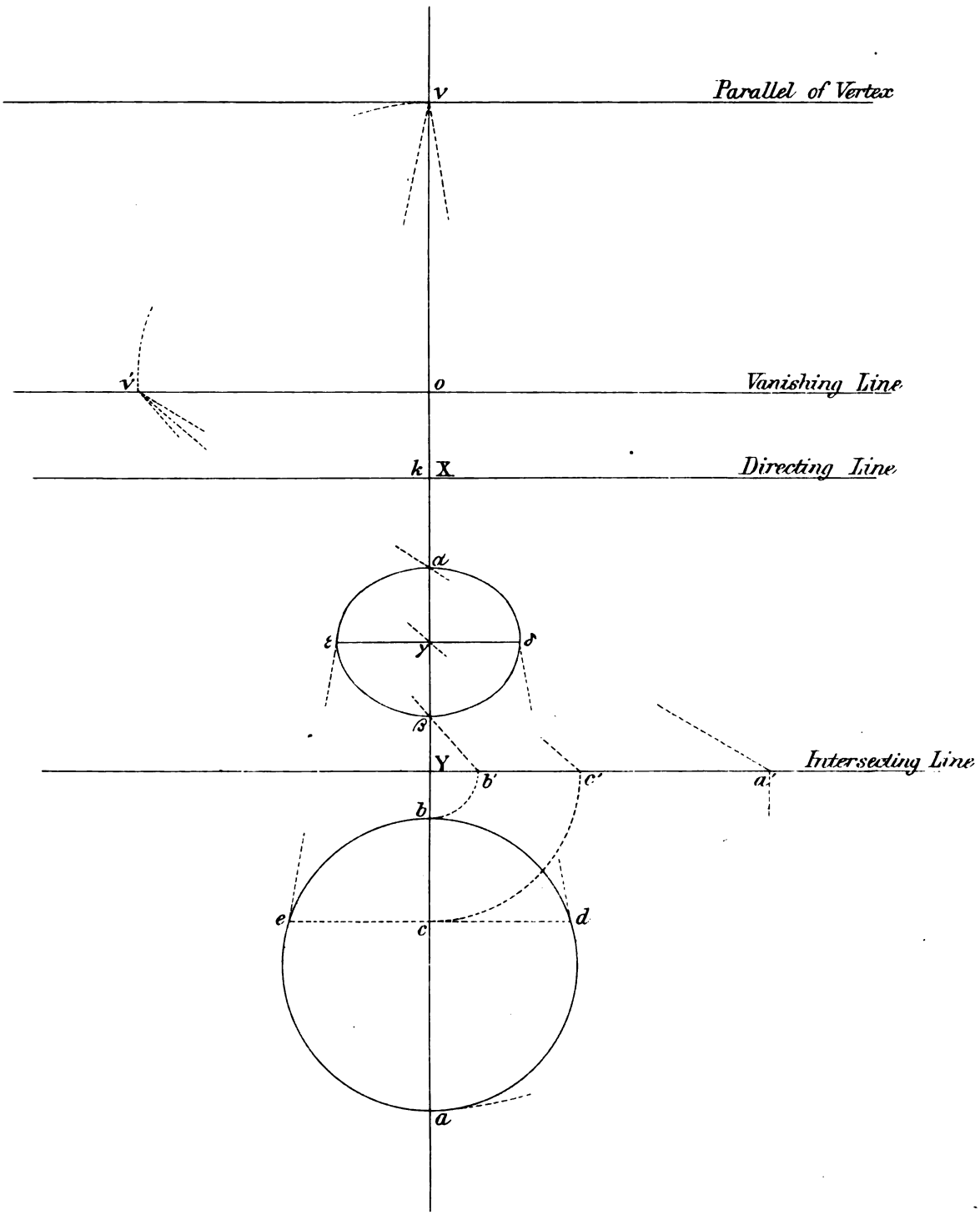
31. If the centre of the original circle falls on the line of station, the construction for finding its perspective is much simplified (Plate VIII.) As before, find $e\,d$, the chord of tangents from k, where $a\,b$ produced cuts the directing line, then $a\,b$ and $e\,d$ are the originals of the axes of the required ellipse, the perspectives of which may be found as shown in the figure, which needs no further description.

If the height of the vertex k Y were equal to $k\,d$ or $k\,e$ the length of the tangent from k to the circle, the perspective would be a circle, as in that case the portion of the cone of projectors cut off by the plane of projection would be similar to the whole original cone, only in a contrary direction.

32. To determine the perspective of a circle which touches the directing line of its plane.

In this case the plane of projection which is parallel to the directing plane will cut the cone of projectors produced in a parabolic curve.

(Plate IX.) Let $a\,g\,e\,b$ be the original circle touching the directing line in a; join $v\,a$ and draw $v\,d$ at right angles to va; with d as a centre and $d\,a$ radius, describe an arc cutting the circle in e; join $d\,e$; bisect the angle $d\,v\,a$ by the line $v\,f$ meeting the directing line in f; from f draw $f\,g$ touching the circle in g, and produce it to meet $a\,e$ produced in k; draw $d\,g$ and produce it to meet $a\,e$ in i, and the circle again in h and join $d\,k$. Then $a\,e$ is the original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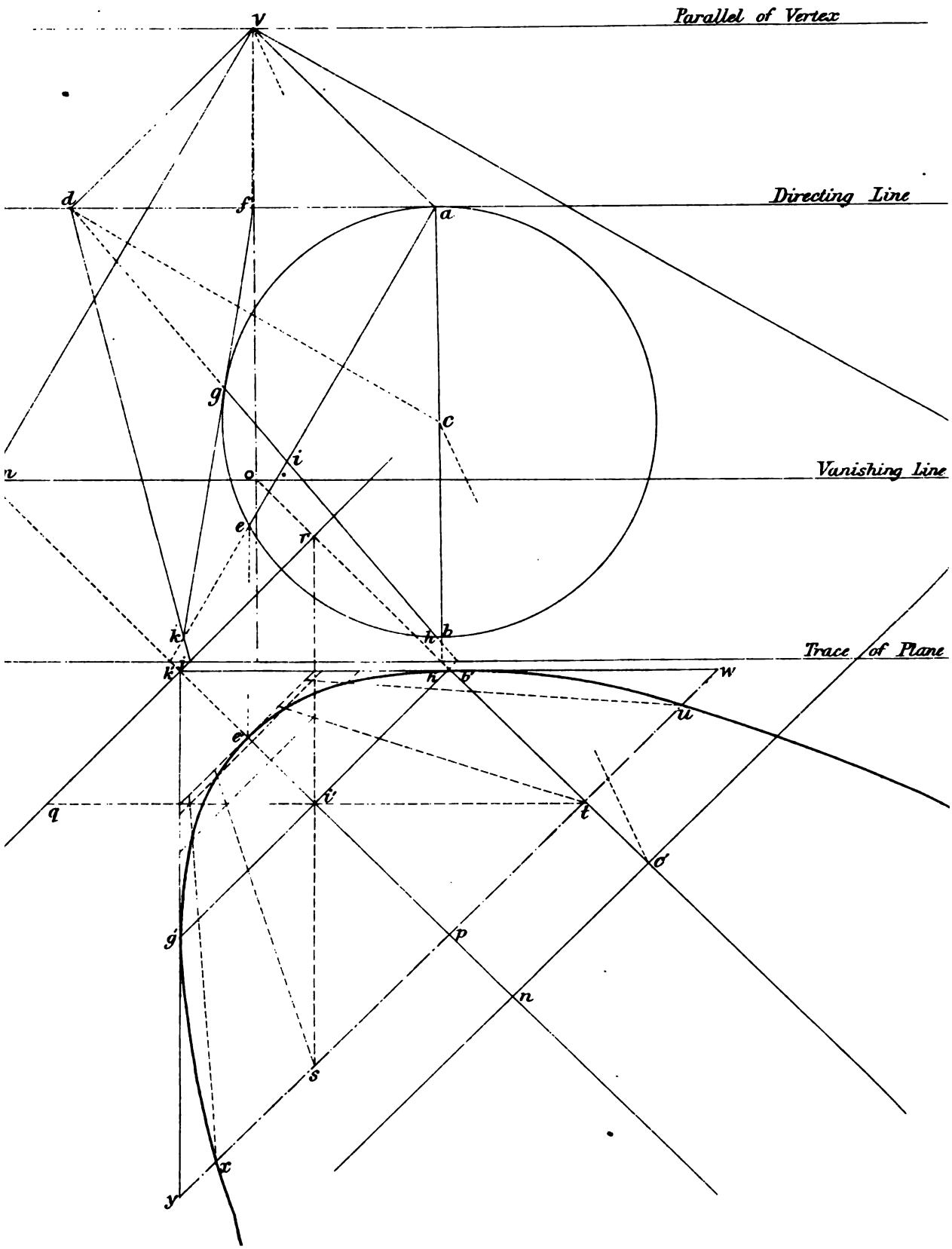
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of the axis of the parabola, and i that of the focus; $g\,h$ is the original of the latus rectum and $d\,k$ that of the directrix.

For, since $a\,e$ is the chord of tangents from d, a point on the directing line, the perspectives of $g\,i$ and $i\,h$ are equal, and similarly the perspectives of all lines within the circle drawn through d will be parallel and bisected by the perspective of $a\,e$; therefore $a\,e$ is the original of a diameter of the parabola, and $g\,h$ that of one of its double ordinates; but because of the directing points d and a, and the angle $d\,v\,a$ being by construction a right angle, therefore the perspectives of $a\,e$ and $g\,h$ are at right angles to each other; therefore $a\,b$ is the original of the axis of the parabola.

Again, because d, f, a are directing points, and the angle $d\,v\,f$ is equal to the angle $f\,v\,a$, therefore the perspective of the angle $i\,g\,k$ is equal to the perspective of the angle $i\,k\,g$; and $f\,g$ being a tangent to the circle at g, the perspective of $f\,g$ will be a tangent to the parabola; therefore $g\,h$ is the original of the latus rectum and i that of the focus, and $d\,k$ that of the directrix, the perspective of $d\,k$ being parallel to that of $g\,h$, both lines having the same directing point d.

The parabola or perspective projection may now be readily deduced from the rules already given, or it may be deduced independently of the original figure in the following manner.

In the first place, the perspective of that diameter of the circle which is perpendicular to its trace or directing line, must have o the centre of the vanishing line for its vanishing point; hence join o and c', the perspective of the centre of the circle, and bisect $o\,c'$ in b', then b' is the vertex of the diameter $b'\,c'$ seeing that the original $b\,a$ is bisected in c, and the perspective of a is at an infinite distance from c'; through c' draw $c'\,l$ at right angles to $o\,c'$ meeting the vanishing line in l. Join $v\,l$ and draw $v\,m$ at right angles to $v\,l$ meeting the vanishing line in m; draw $m\,n$ parallel to $b'\,c'$ and cutting $l\,c'$ in n; bisect $n\,m$ in e'; then e' is the vertex and $e'\,n$ the axis of the parabola; through b' draw a tangent to the parabola (which is readily done, for since the tangent to the circle at b is parallel to the directing line, its perspective will be parallel to the vanishing line), cutting $n\,e'$ in j; bisect $j\,b$ by a perpendicular cutting $n\,e'$ in i', then i' is the focus; through i' draw $g'\,h'$ making $i'\,h', i'\,g'$ each equal to twice $i\,e'$, and $g'\,h'$ is the latus rectum; lastly, make $e'\,k'$ equal to $e'\,i$ and draw $q\,r$ through k' perpendicular to $k'\,n$.

A simple method of finding points in the parabola in order to draw the curve is the following:

Make $i' p$ equal to $i' k'$, and draw $y p w$ parallel to the directrix; make $k' r$, $k' q$, each equal to half the latus rectum; join $r i', q i'$, and produce to meet $y w$ in s and t, and join $k' g', k' h'$, tangents to the parabola at g' and h', and produce to meet $y w$ in y and w; in $t w$ find a point u, such that $t u$ is to $t w$ as 11 is to 15, similarly for x between s and y, and join $k' x, k' u$; through the points where $r s$ or $q t$ cuts the lines radiating to k', draw lines parallel to the directrix, and the curve drawn through the points of intersection as shown in the figure will be the curve required.

It is worthy of notice that the tangents to the circle from v will also be tangents to the parabola from the same point.

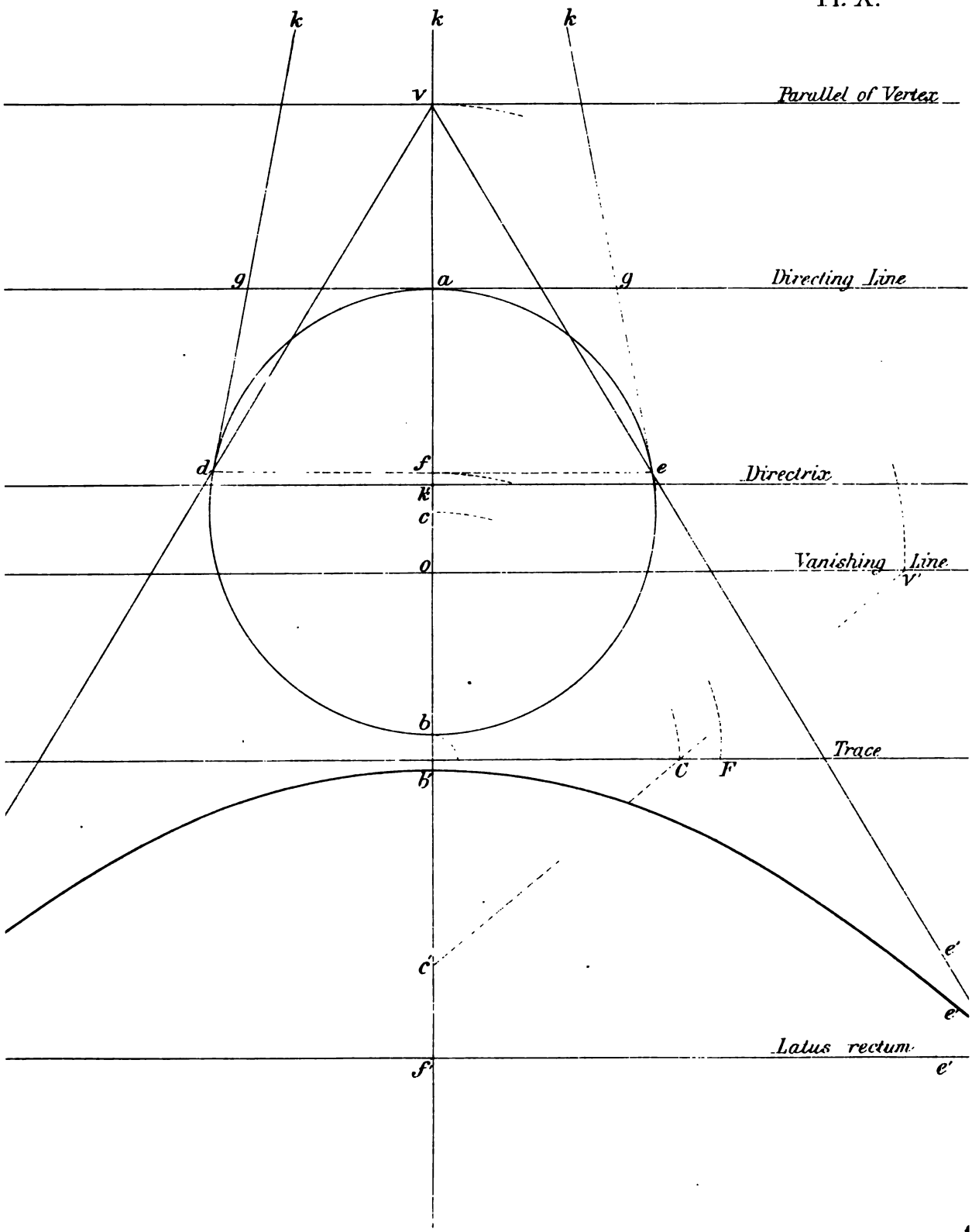
33. When the centre of the circle lies on the line of station and touches the directing line, its perspective may be found in the following manner:

(Plate X.) From a the point of contact, step out $a g$ on the directing line equal to $a v$, the height of the vertex; through g draw $g e$, a tangent to the circle in the point e, and produce it to meet the line of station $b a$ produced in k; draw the chord $d e$ parallel to the directing line. Now, since the perspective of the triangle $k f e$ is evidently similar to the triangle $v a g$; therefore the perspective of $k e$ makes equal angles with the perspective of $f e$ and $a b$ and hence the perspectives of $f e$ and $f k$ are equal; it follows that $d e$ is the original of the latus rectum; and a line through k parallel to the directing line, the original of the directrix; and f, that of the focus; and b the original of the vertex of the required parabola, which can be described without further difficulty as indicated in the figure.

34. The last case is that in which the original circle cuts the directing line of its plane in two points d, e.

(Plate XI.). Draw the tangents $d s, e s$ meeting the perpendicular diameter $a b$ produced in s; since d and e are directing points their perspectives are infinitely distant, and therefore the perspectives of the tangents at these points will be tangents to the perspective of the circle at an infinite distance; but in this case it is known that the perspective of the circle is an hyperbolic curve, or two opposite hyperbolas, therefore $s d$, $s e$ are the originals of the asymptotes of these hyperbolas, and s, the point where they intersect, is the original of the centre. Join $v d$, $v e$, and bisect the angle $d v e$ by the line $v f$, cutting the directing line in f, and draw $s f$, cutting the circle in g and h; from v draw $v k$ perpendicular to $v f$ to meet the directing line in k, join $k s$, and produce $g d, g e$,

Pl. X.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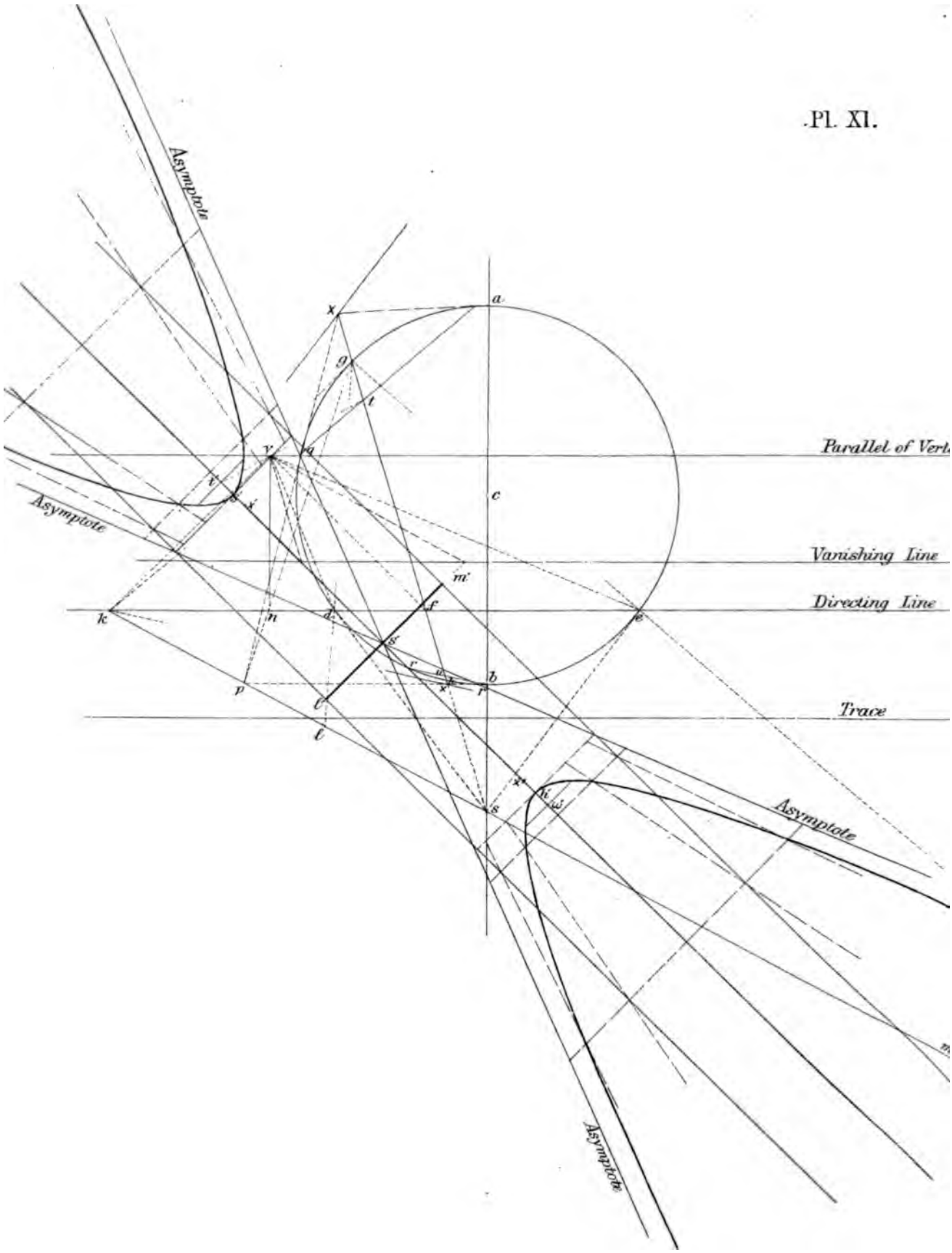
Pl. XI.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to meet $k\ s$ in l and m. Now, because $d\ e$ is the chord of the tangents from s, any line as $s\ h\ f\ g$ will be harmonically divided in the points s, h, f, g, and f being constantly a directing point, the perspectives of g and h will be at an equal distance from the perspective s; and s being the original of the centre of the hyperbolas, $h\ g$ is the original of a diameter. Again, because $d\ v\ e$ is bisected by $v\ f$ to which $v\ k$ is drawn perpendicular, therefore $k\ e$ is harmonically divided in k, d, f, e; and hence, f is a point in the chord of tangents from k; but $d\ e$ being the chord of tangents from s, and k being a point in this chord, the chord of the tangents from k must pass through s, therefore $g\ h$, which passes through f and s, must be that chord, and consequently the perspectives of all lines drawn through k and terminated by the circle are bisected by the perspective of $g\ h$, and hence all such lines are double ordinates to the diameter represented by $g\ h\ s$; but since the angle $k\ v\ f$ is a right angle, the perspectives of all lines drawn through k, are perpendicular to the perspective of $g\ h$, and therefore $g\ h\ s$ is the original of the major axis of the hyperbola and $k\ s$ that of the minor axis, and g and h are the originals of the opposite vertices of the opposite sections; because of the directing points d, e, the perspectives of the lines $g\ d$, $g\ e$ will be parallel to the perspectives of $d\ s$, $e\ s$, or will be lines drawn through the vertex of the hyperbola parallel to the asymptotes, and hence the points l and m in which $g\ d$, $g\ e$ produced, cut the original of the minor axis $k\ s$, will be the originals of the extremities of the minor axis; f is the directing point of all ordinates to the minor axis, and the angle $d\ v\ e$ is the angle between the asymptotes of the hyperbola. To find the originals of the foci, bisect the angle $k\ v\ f$ by the line $v\ n$ cutting the directing line in n, and through n and either of the extremities of the original of the major axis as g, draw $g\ n$ and produce till it cuts the original of the minor axis $l\ s\ m$ in p; from p draw tangents to the circle touching it in q and r, and produce them to meet $g\ h$ produced in x, x; join $k\ x$, $k\ q\ t\ a$, $k\ r\ a\ r$, and $k\ x$; then the points t and u in which $k\ q$ and $k\ r$ cut $g\ h$ will be the originals of the foci, and $k\ x$ the original of the directrix.

Having thus completed the original lines of the principal parts of the section, the section or double hyperbola may be readily deduced by the methods already fully explained and further shown in the figure.

35. This projection may be determined, having given the perspective $a\ o\ b$ of that diameter of the circle which is perpendicular to the directing line, without reference to the original plane in the following manner (Plate XII.) Bisect $a\ b$ in s, the centre of the hyperbola. Since $a\ b$ is perpendicular to the

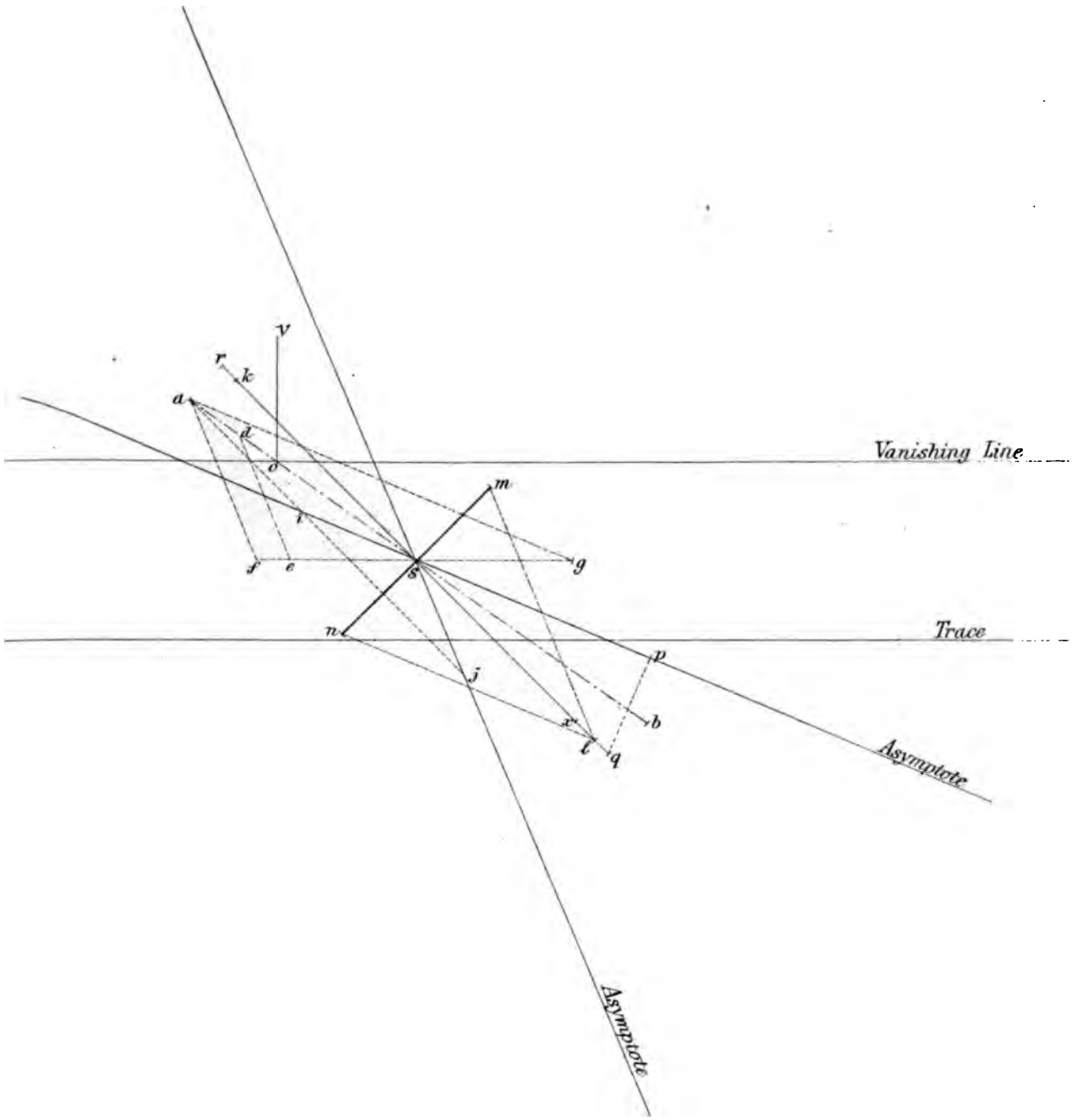
directing line, the tangent to the circle at the original of a or b is parallel to the plane of projection, and therefore the conjugate diameter to $a\ b$ is a line through s parallel to the vanishing line. In this line make $s\ e$ equal to $o\ v$, the distance of the vertex, and in $s\ a$, make $s\ d$ a mean proportional to $o\ a$ and $o\ b$; join $d\ e$ and draw $a\ f$ parallel to $d\ e$ to meet $s\ e$ in f; then $f\ s\ g$ is the conjugate diameter to $a\ s\ b$. Join $a\ g$ and through s draw lines parallel to $a\ f$ and $a\ g$; these will be the asymptotes of the proposed hyperbolas.

Bisect the angles between the asymptotes by the lines $k\ l$, $m\ n$, which will give the directions of the principal axes. Through a draw $a\ i\ j$ parallel to $k\ l$ to cut the asymptotes in i and j, and make $s\ k$, $s\ l$, a mean proportional to $a\ i$, $a\ j$. Then $k\ l$ is the major axis, and by drawing $l\ m$, $l\ n$, parallel to the asymptotes, we obtain $m\ n$ the minor or conjugate axis. The foci may be found by making $s\ p$ on the asymptote equal to $s\ l$ the semi-major axis, and $p\ q$ perpendicular to the asymptote, will cut $k\ l$ produced in q, the required focus. The latus rectum is a third proportional to the major and minor axes; and the foot of the directrix x may be found by making $s\ q : s\ l :: q\ l : l\ x$.

36. When the centre of the original circle lies in the line of station, but not in the station point or foot of the director of the vertex, as in Plate XIII., the original of the centre of the hyperbolas is found, as before, by drawing tangents to the circle through l and m, where the circle cuts the directing line; in this case these tangents will meet in s, a point in the line of station; $a\ b$ the perpendicular diameter of the circle, is the original of the transverse axis, and a line $c\ d$ through s, parallel to the vanishing line, will be the original of the conjugate axis, and its magnitude is determined, as before, by drawing $m\ b$, $l\ b$, or $a\ l$, $a\ m$, to meet it in c and d. Again, by making $y\ f$ equal to $y\ v$, and through f drawing tangents $f\ i$, $f\ j$ to the circle, the chords $i\ g\ i$, $j\ h\ j$, drawn parallel to the directing line, will be the original of the latus rectum in each hyperbola, and g and h their respective foci. From these data the perspective projection may be readily deduced; only one of the hyperbolas has been shown in the plate to avoid confusion.

37. When the centre of the original circle lies in the directing line and in the line of station, as in Plate XIV., it is obvious that the tangents to the circle at the points l, m, in which the directing line cuts the circle, being parallel, will never meet, and therefore the original of the centre of the hyperbolas has no actual existence; but as these tangents are perpendicular to the directing line,

Pl. XII.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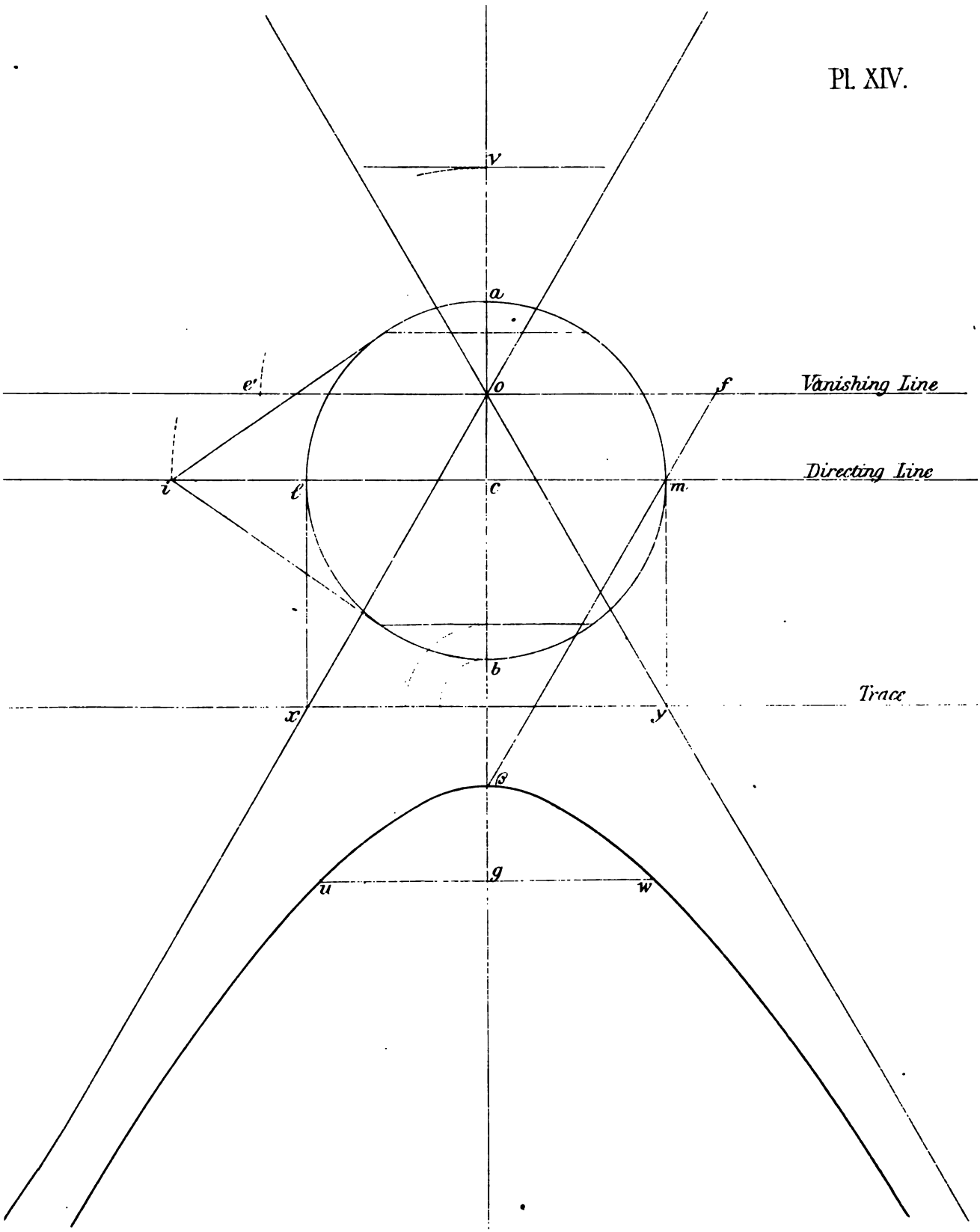
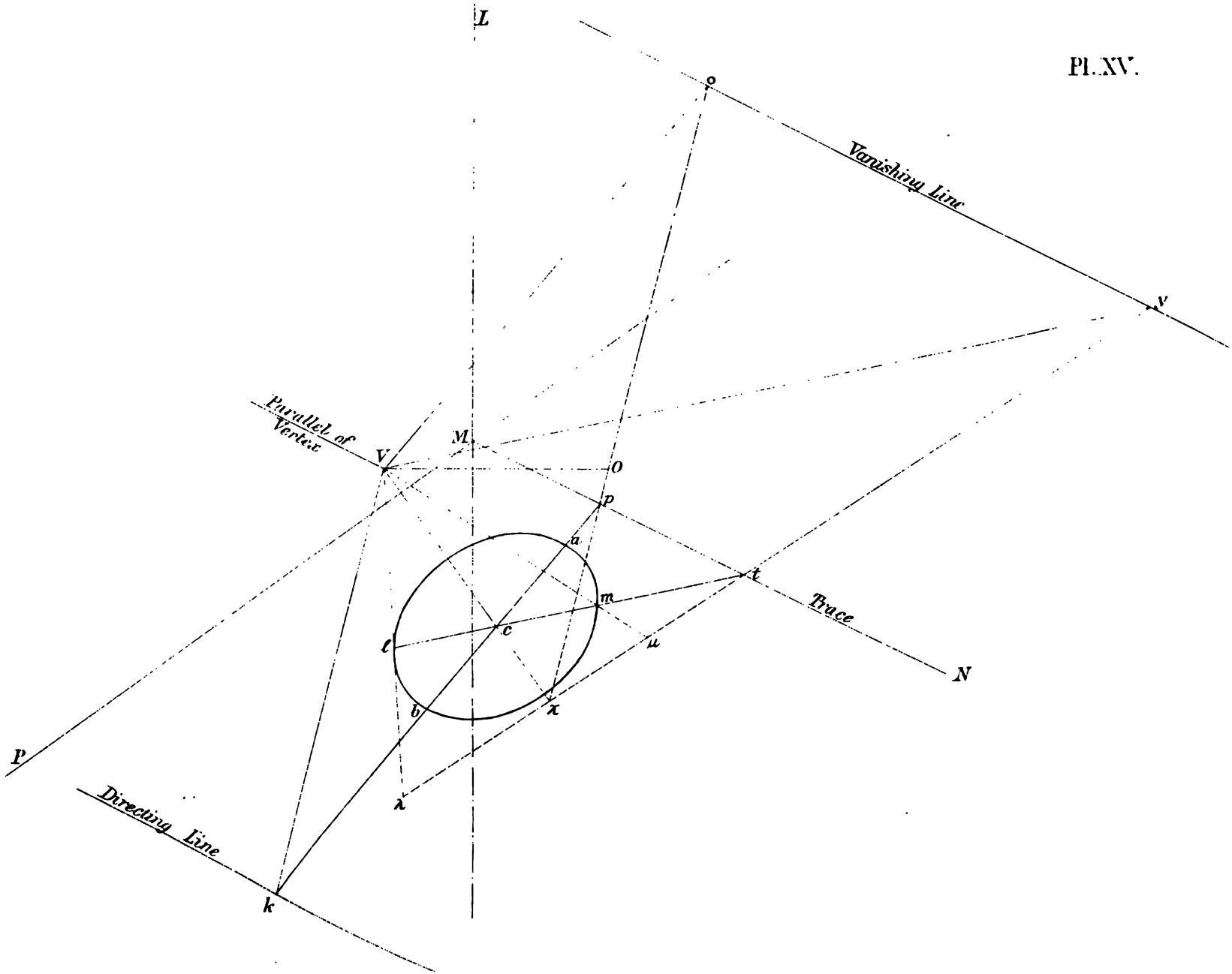
Pl. XIV.

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their perspectives or the asymptotes of the hyperbolas, must have o, the centre of the vanishing line, for their vanishing point. Therefore o is the centre of the hyperbolas, and lines drawn through o and x, y, the traces of the tangents $l\,x$, $l\,y$, will be the asymptotes required. The remainder of the construction can be readily deduced from the figure.

38. We will proceed to show how the preceding projections of a circle may be made useful in determining the properties and generation of vanishing lines and points.

(Plate XV.) Let L M N be the plane of projection meeting the plane of the paper in L M, and let P M N be any given original plane meeting the plane of the paper in P M and the plane of projection in M N; then M N is the trace of the original plane. Let V be the vertex situated out of the paper. Draw V c perpendicular to the original plane meeting it in c, and produce it to meet the plane of projection in x; also draw V O perpendicular to the plane of projection meeting it in O, the centre of the plane; join x O, cutting the trace M N at right angles in p; through V draw V k parallel to x O to meet $p\,c$ in k; then k is the point of station, and a line drawn through k parallel to the trace will be the directing line; through V draw a line V o parallel to $k\,p$ to meet x O in o; then o is the centre of the vanishing line of the original plane. Since x is the vanishing point of all perpendiculars to the original plane, it follows that the vanishing lines of all planes perpendicular to the original plane must pass through x. If it were required to determine the locus of the vanishing points of all lines which incline to the original plane in any given angle, it would be necessary to describe a right cone on the original plane with vertex V, axis V c and generating line V m, making the angle V $m\,c$ equal to the given angle; then it is evident the required locus will be the projection of the circular base of the cone, since the radial of any line inclined at the given angle to the original plane must coincide with one of the sides of the cone. If the given angle of the cone V $b\,c$ be greater than the angle V $k\,c$ or the angle of inclination between the original plane and the plane of projection, the locus required will be an ellipse; if the angle V $b\,c$ be equal to the angle V $k\,c$, or if the cone touches the directing plane along the director V k the required locus will be a parabola; and lastly, if the angle V $b\,c$ be less than the angle V $k\,c$, the circle will cut the directing line, and hence the required locus will be two opposite hyperbolas; all these several projections of the forming circle have been fully discussed in the preceding Articles. To determine the perspective of any given diameter of the circle

as $l\,c\,m$, draw radial V y of the diameter $l\,m$, meeting the vanishing line of the original plane in y, and join $y\,x$; then $x\,y$ will be the vanishing line of planes perpendicular to the original plane and parallel to the line $l\,m$, and since V c is perpendicular to V y and $l\,m$, the radials of the extremities of the given diameter, V m, V l must be inclined to V y, the radial of the diameter, at angles equal to the angle V $m\,c$ or V $l\,c$. If it be required to find the vanishing lines of planes inclined to the original plane at a given angle, it is evident the planes through the vertex producing such vanishing lines must all be tangential to the cone inclined at the angle required; and therefore the vanishing lines themselves must be tangents to the projection of the circular base of the cone. Hence it follows that the vanishing lines of all planes inclined to an original plane at a given angle will trace out on the plane of projection a curve which will be the projection of the circular base of the cone on the original plane.

39. Before discussing some problems respecting the generation of vanishing lines, the following construction will be found useful:—Having given the projection of a diameter of a circle, to find the projection of any other diameter.

(Plate XVI., Fig. 1.) Let $a\,b$ be the given projection; produce $a\,b$ to meet the vanishing line $w\,a$ in u; assume any other two vanishing points, w, x, so that the radials $v'\,w$, $v'\,x$, shall be perpendicular to each other. Then the points c and d in which the lines $w\,a$, $w\,b$, $x\,a$, $x\,b$, intersect each other are points on the projection of the circle, because the originals of the angles $w\,c\,x$, or its equal $a\,c\,b$ and $w\,d\,x$ are right angles, and $a\,b$ is a diameter by hypothesis. Again, because the originals of $x\,a\,d$, $c\,b\,d$, are also right angles, $c\,d$ must be the projection of a diameter of the circle, and therefore s, the intersection of $a\,b$ and $c\,d$, must be the projection of the centre of the circle. Again, since the originals of the angles $c\,a\,b$, $c\,d\,b$, are equal, if $d\,c$ be produced to its vanishing point y, the angle between the radials $v'\,y$ and $v'\,x$ must equal that between $v'\,x$ and $v'\,u$. Hence, if the projection of one diameter $a\,b$ be given, and the direction of a second $d\,c$, its extremities may readily be found by bisecting the angle between their radials $y\,v'\,u$ by the radial $v\,x$, and lines drawn from x to a and b will cut $c\,d$ in the points required.

We are now in a position to discuss the following problem.

40. Having given the vanishing lines of planes at right angles to each other, to find the vanishing points of lines lying in one of the given set of planes and inclined at a given angle to the other given set. (Plate XVI. Fig. 2.)

CASE 1. When one of the sets of planes is perpendicular to the plane of projection, and consequently their vanishing line as L M must pass through the centre O of the plane of projection. In this case the vanishing line of planes P Q must be perpendicular to L M.

Make O v' = O V, the distance of the vertex, and make the angles O v' a, O v' b, equal to the given angle, then a b is the projection of the perpendicular diameter of the forming circle in the planes L M, which is the base of the cone formed by lines drawn through the vertex inclined to the planes L M at the given angle; in this case the locus of the lines so inclined, which is the projection of the forming circle, must be two opposite hyperbolas, of which O is the centre, a b the transverse axis, and v' o half the conjugate axis; hence P Q, the vanishing line of the second given set of planes, must be the direction of the projection of a diameter of the forming circle, and its extremities may be found by the preceding problem, by bisecting the angle x V O by the line V w, and then producing the lines a w, b w, to cut P Q in y z. Therefore, these points are the vanishing points of the lines required.

A much simpler solution in this case is the following: Make x v'' equal to the radial x v, and make the angles x v'' y, x v'' z, each equal to the given angle. Then y and z are the points required, x being the vanishing point of the perpendicular seats of all lines in the planes P Q to the planes L M.

CASE 2. When neither vanishing line passes through the centre O.

Since the planes L M and P Q are perpendicular to each other, the vanishing line P Q must contain the vanishing point of lines perpendicular to the planes L M. Hence the vertical line O o of planes L M must pass through x the vanishing point of perpendiculars, to planes L M. On o x describe a semicircle, and through O draw O V at right angles to o x and intersecting the circle in V; then O V will be the distance of the vertex. Join V o and make the angles o V a, o V b, each equal to the given angle, cutting the vertical line O o in a and b; a b is the projection of the perpendicular diameter of the forming circle, and x being the vanishing point of perpendiculars to the planes L M, must be the projection of the centre of the forming circle, and therefore the vanishing line Q P which passes through x must be the direction of the projection of a second diameter of the forming circle, the extremities of which may be found as before; viz. by making o v equal to the distance V o of the planes L M, bisecting the angle o v z by the line v u, and the lines joining a, u, and b, u, will cut P Q in the points required, y and w.

When the angle $a\,V\,o$ which the lines are inclined at to the planes L M is less than the angle V o O (which is the angle of inclination of the planes L M with the plane of projection), it is obvious the forming circle will cut the directing line, and, hence its projection, which is the required locus, will be two opposite hyperbolas having $a\,b$ for its transverse axis, and a perpendicular through s, the centre of $a\,b$, will be the direction of its conjugate axis, the length of which may be readily found, as in the figure, by making $e\,s$ and $s\,f$ in the same ratio to V o as $s\,a$ or $s\,b$ is to a mean proportional between the segments $o\,a$, $o\,b$.

When the angle of inclination of the required lines to the planes L M is equal to the angle of inclination of these planes to the plane of projection, or when the angle $a\,V\,o$ (Plate XVIII.) is equal to the angle V o O, the projection of the forming circle or locus required is a parabola; x the vanishing point of perpendiculars, is the projection of the centre, b that of the extremity of the perpendicular diameter of the circle and the vertex of the parabola; the projection of the other extremity has no existence, because V a is in this case parallel to $x\,o$; but a line drawn through x parallel to L M will be the projection of a diameter of the circle parallel to the plane of projection, and its extremities are determined by making $x\,r$, $x\,s$, each equal to V o, the distance of the planes L M. As before, the vanishing points required on the line P Q may be determined, by bisecting the angle $o\,v\,z$ by the line $v\,u$, and joining $u\,b$ to meet the line P Q in y; but as the point d may be considered to be infinitely distant, the line through u parallel to $x\,o$ will cut P Q in w, the other vanishing point.

The vanishing points in this, as in the first case, may, also more simply, be found by making angles equal to the given angle, with the radials of z, (V′ z); V′ O o' being the radial of the planes P Q.

Lastly, when the angle of inclination of the required lines to the planes L M is greater than the angle of inclination of these planes to the plane of projection, or when the angle $b\,V\,o$ (Plate XIX.) is greater than the angle V o O, the projection of the forming circle or locus required is an ellipse, the major axis of which, $a\,b$, will have one extremity b on $x\,o$ below x and the other a between x and o; x being the projection of the centre of the forming circle, $r\,x\,s$ drawn parallel to L M, will be the direction of the projection of the diameter of the circle, which is parallel to the directing line, and its magnitude is determined by joining $v'\,a$ to cut $x\,s$ in s. The minor axis of the ellipse, $c\,d$, may be determined by making $a\,e$ equal to a mean proportional between $a\,x$ and $x\,b$, and drawing $s\,f$ parallel to $a\,b$ and $e\,f$ parallel to $r\,s$, meeting in f; bisect $a\,b$ by the perpendicular $c\,d$, join $a\,f$ to meet $c\,d$ in d, one extremity of the minor

Fig. 1.

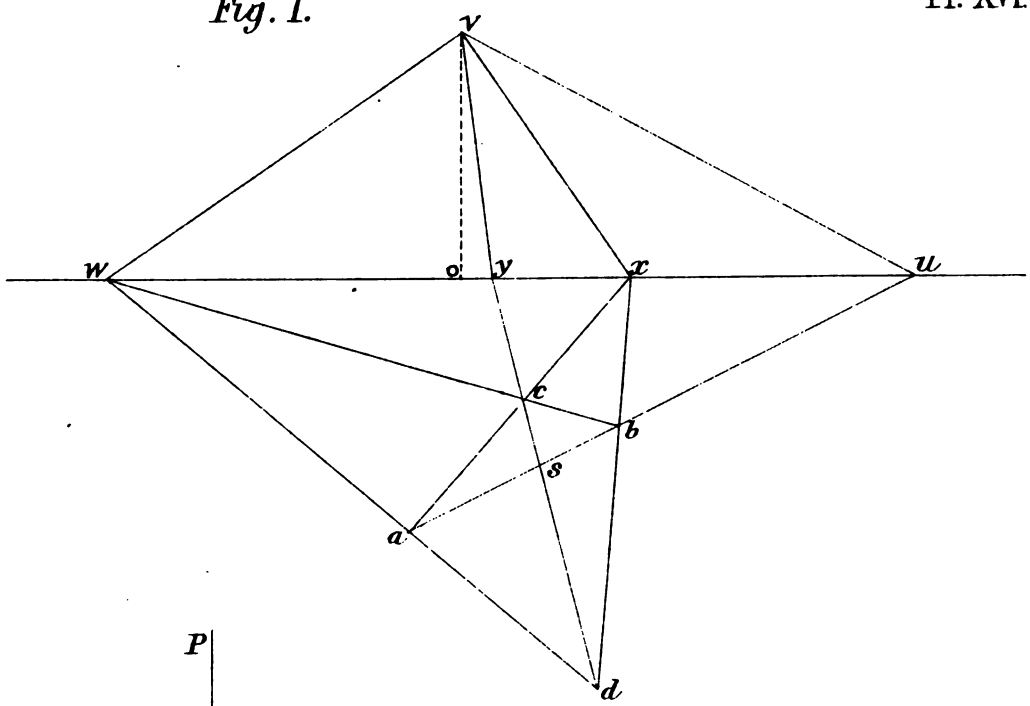
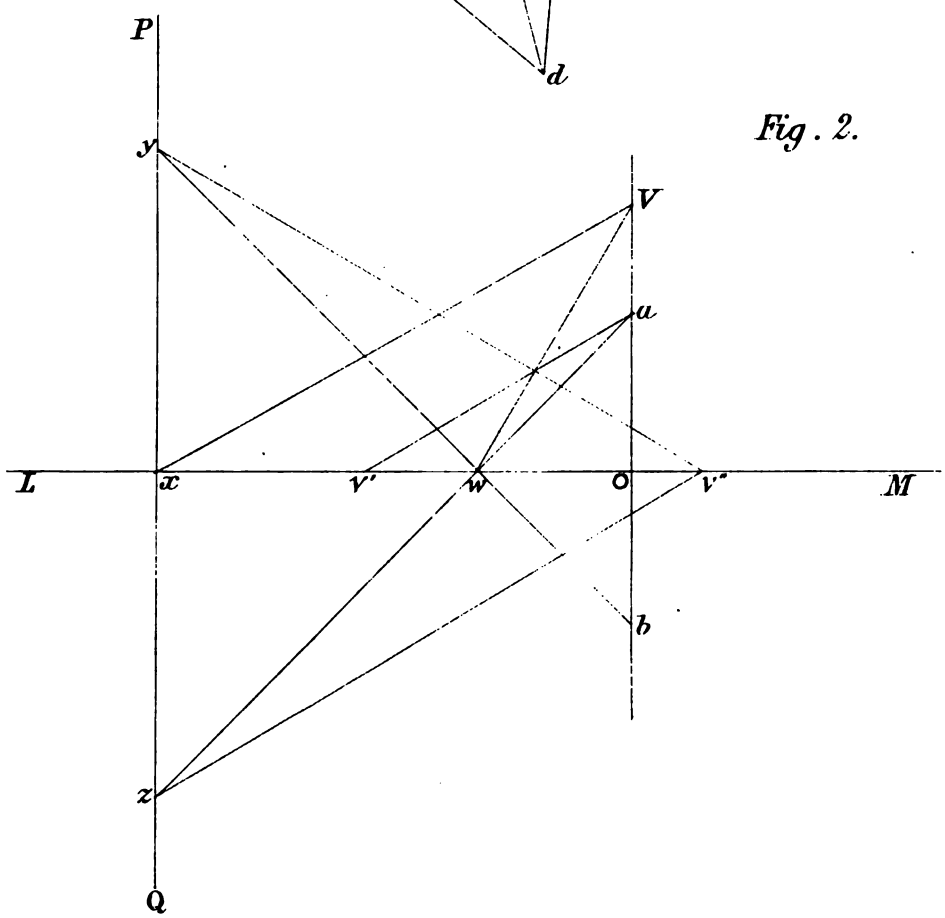
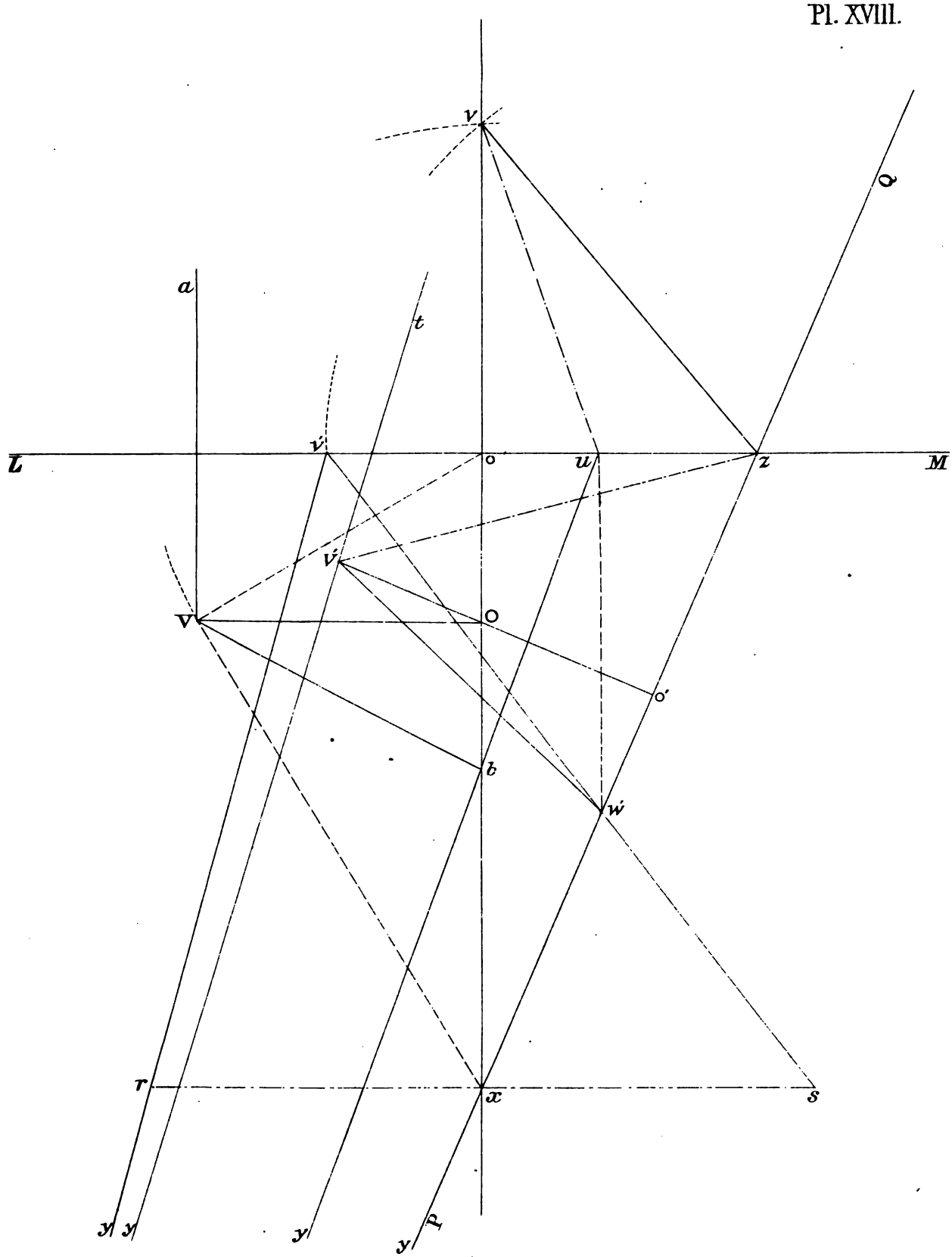


Fig. 2.

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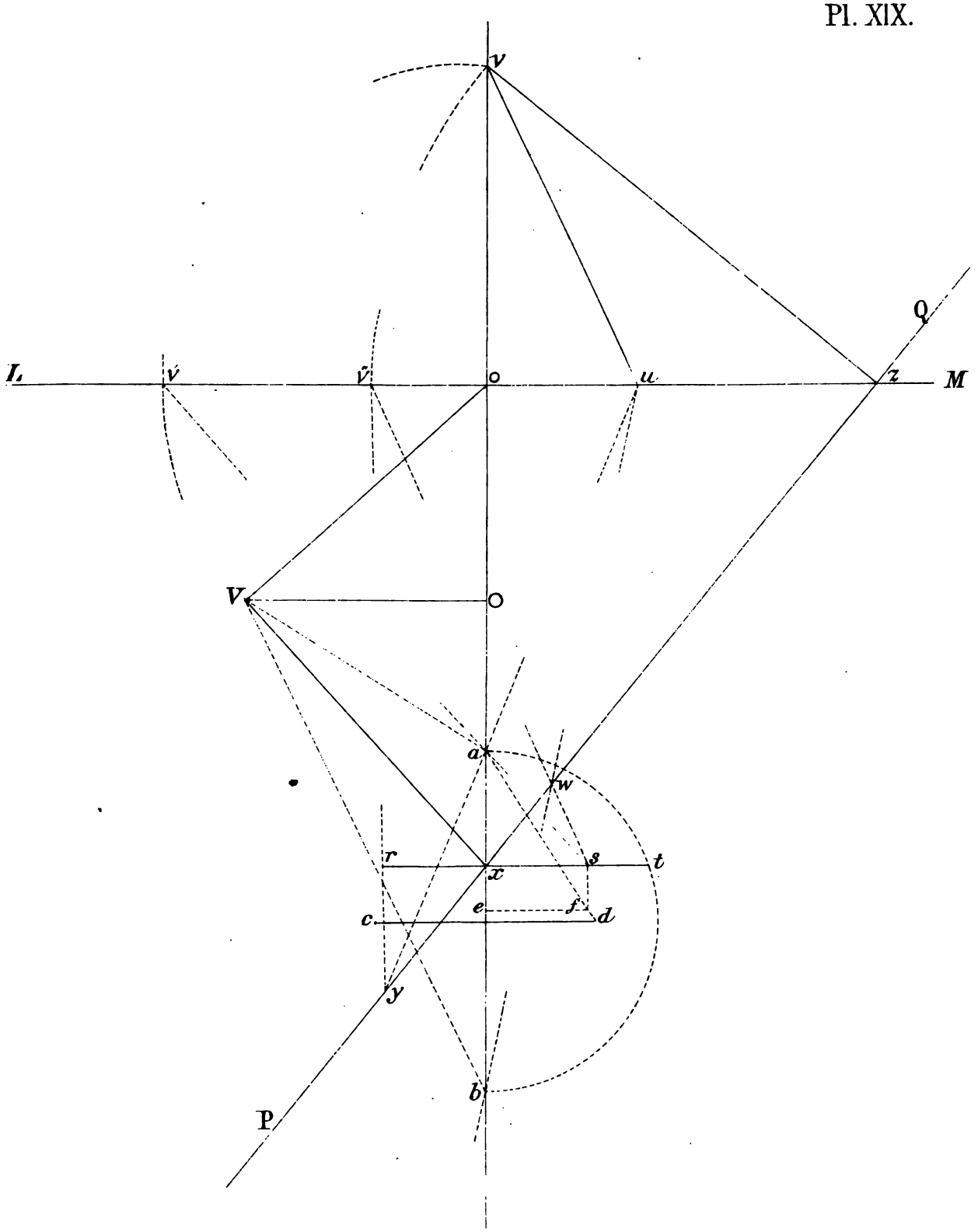
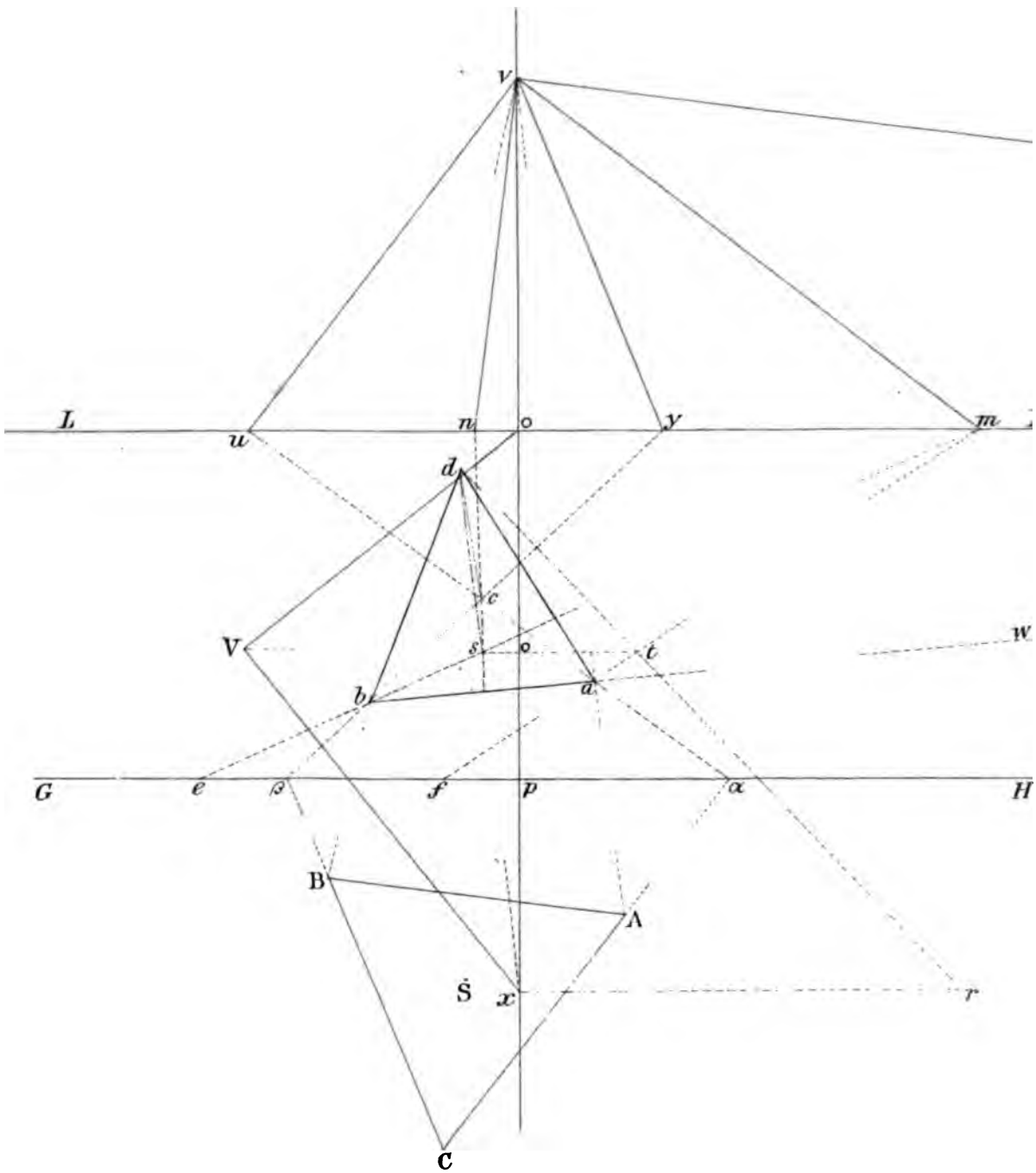
Pl. XIX.

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axis. The vanishing points y, w, on any vanishing line P x Q inclined to planes L M as required, may be readily determined by either of the methods shown in the other cases; viz. by the lines joining v'' and $r\,s$, or by those joining u and $a\,b$.

41. To determine the projection of a tetrahedron, having given the inclination of one of its faces A B C. (Plate XX.)

Determine the plane of A B C by its vanishing and intersecting lines L M and G H, and having found the projection of the face $a\,b\,c$ by means of the radials $v\,a$, $v\,y$ and $v\,w$ of its sides A C, C B and B A and their respective traces, determine the projection of the centre of this face. This may be done by finding the vanishing point of lines perpendicular to A C. This is readily done by drawing the radial $v\,m$ perpendicular to $v\,u$ the radial of A C, and the line joining b and m will pass through the centre required. In like manner, draw $v\,n$ perpendicular to $v\,w$, the radial of A B and join $c\,n$ and produce it to cut $b\,m$ in s the projection of the centre. Join s and x, the vanishing point of perpendiculars to the planes L M. Then $s\,d$ will be the direction of the axis of the solid, and it only remains to determine its height. This may be done, as shown in the projection of a cube, by the following construction. Produce $m\,s\,b$ to meet G H in e, make $e\,f$, along G H, equal to the length of the axis, and draw $s\,t$ parallel to G H; join $f\,m$, meeting $s\,t$ in t; through x draw $x\,r$ parallel to G H and equal to the radius x V; join $r\,t$ and produce it to meet $x\,s$ produced in d the projection of the vertex of the solid. Lastly, join $d\,a$, $d\,b$, $d\,c$, and the projection will be completed.

42. Another method of solving this problem is by means of the vanishing planes of the several faces. (Plate XXI.)

Having determined the projection $a\,b\,c$ of the given face A B C, lying in the given plane (L M, G H) as in the preceding Article, draw $e\,l$ parallel to the radial V o, and draw V l perpendicular to $e\,l$, and meeting it in l; make the angle V $f\,l$ equal to the angle of inclination between the faces of a tetrahedron; then, in the vertical line $e\,o$ make $e\,l'$ equal to $e\,l$; and with l' as centre and $l\,f$ as radius describe a circle, and draw three tangents to this circle parallel to the sides A B, B C, and C A. These tangents represent the intersections of the vanishing planes with the plane A B C, when turned down into the paper about its trace G H, and the points p, p' in which these tangents meet the trace, will not move when the plane is turned back into its original position; therefore, these points will be points in the vanishing lines of the required planes; but the vanishing points P and P_1 of the lines $b\,c$ and $a\,c$ must also be points in the re-

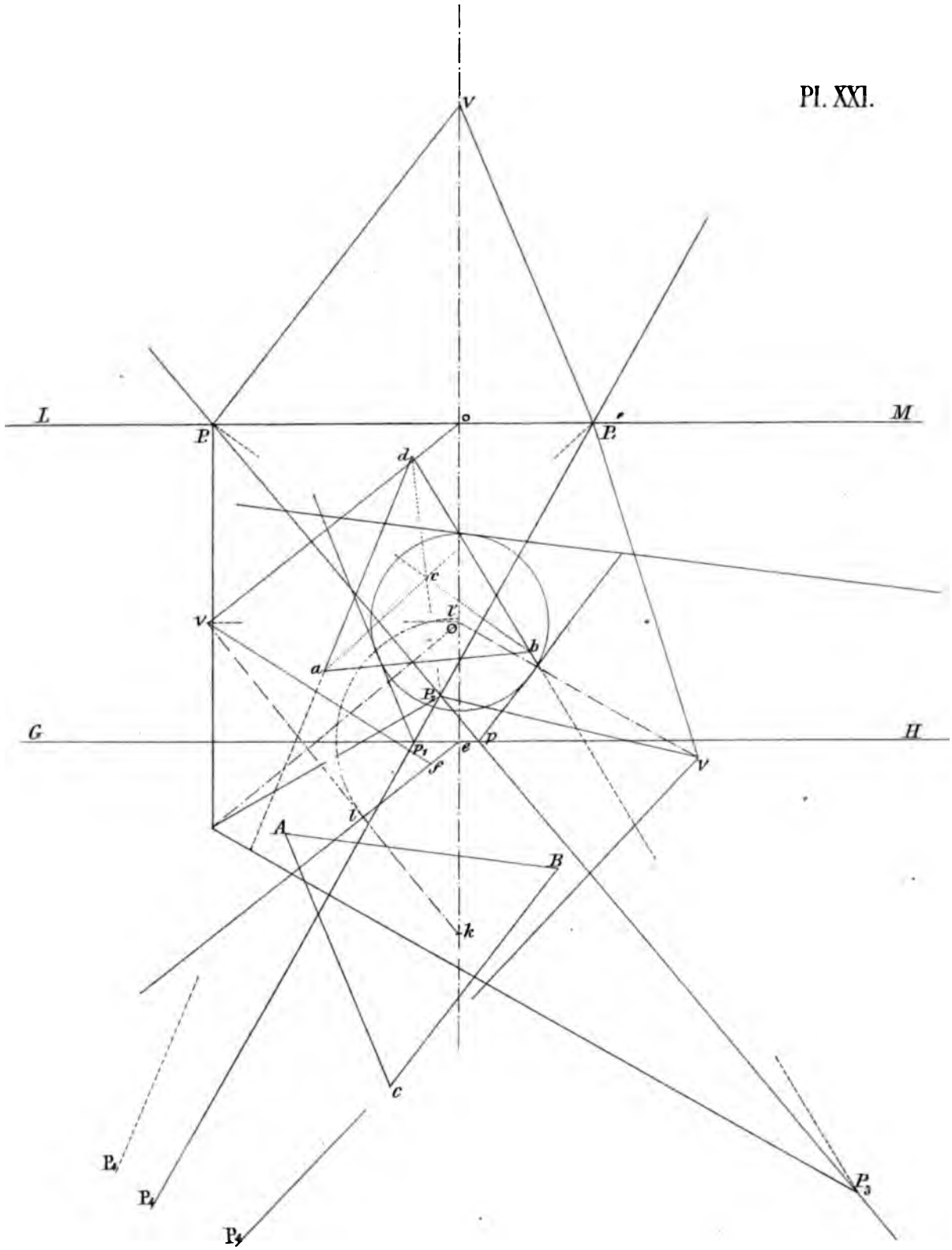
quired vanishing lines; hence P p, P_1 p_1 are two of the vanishing lines sought; the intersection of the third tangent with G H is too remote to allow the vanishing line of the remaining face to be drawn. The point P_2, in which the vanishing lines p P, p_1 P_1 intersect, is evidently the vanishing point of the intersections of the planes p P, p_1 P_1; therefore, the line joining P_2 and c, produced, will be the direction of one of the edges of the solid. Lastly, on the vanishing lines p P, p_1 P_1, find the vanishing points P_3, P_4, which make, with the vanishing point P_2, angles equal to 60°, the inclination of the edges of the solid to each other;—and lines drawn through P_3 b, P_4 a, will meet P_2 c, in d, the vertex of the tetrahedron; and the solid will be completed.

43. Having given the projections a, b, c, of three points, and the lengths of their respective projectors, (or the distances of the original points from the vertex) to determine the plane containing the three points, by its vanishing line and trace.

Join a b, and through O, the centre of the plane of projection, draw O o v perpendicular to a b. Draw O V, the distance of the vertex, parallel to a b, and make o v in O o produced equal to o V; or more shortly, make o l, in a b produced, equal to the distance of the vertex, and step out o v equal to O l. Join v a, v b, and produce to A and B, making v A, v B, equal to the given projectors of a, b. Thus, the projecting plane V A B will have been turned down about a b until it coincides with the plane of projection; therefore, if A B be produced to meet a b produced in t, and v p be drawn parallel to A B, to meet a b produced in p, t will be the trace, and p the vanishing point of the line a b; in like manner, find t', t'' and p', p'', the traces and vanishing points of a c and b c respectively; and a line drawn through p p' p'' will be the vanishing line of the required plane, and t t' t'' its trace. It is obvious, also, that the lines A B, B C, C A, are the respective real lengths of a b, b c, c a. Lastly, if through the points A, A, A α, A α be drawn perpendicular to a b, a c, the point α, in which they intersect, will be the orthographic projection of A on the plane of projection, and will obviously lie on the line O a produced. Similarly with the points β, γ.

Lastly, the traces of a β, a γ, β γ, will obviously lie on the trace of the plane of A B C, and be the points t, t' and t'' already found.

44. To draw a perpendicular to a given plane (L M, G H) (Plate XXIII.) from a given point p outside the plane, and to find the point of intersection.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L
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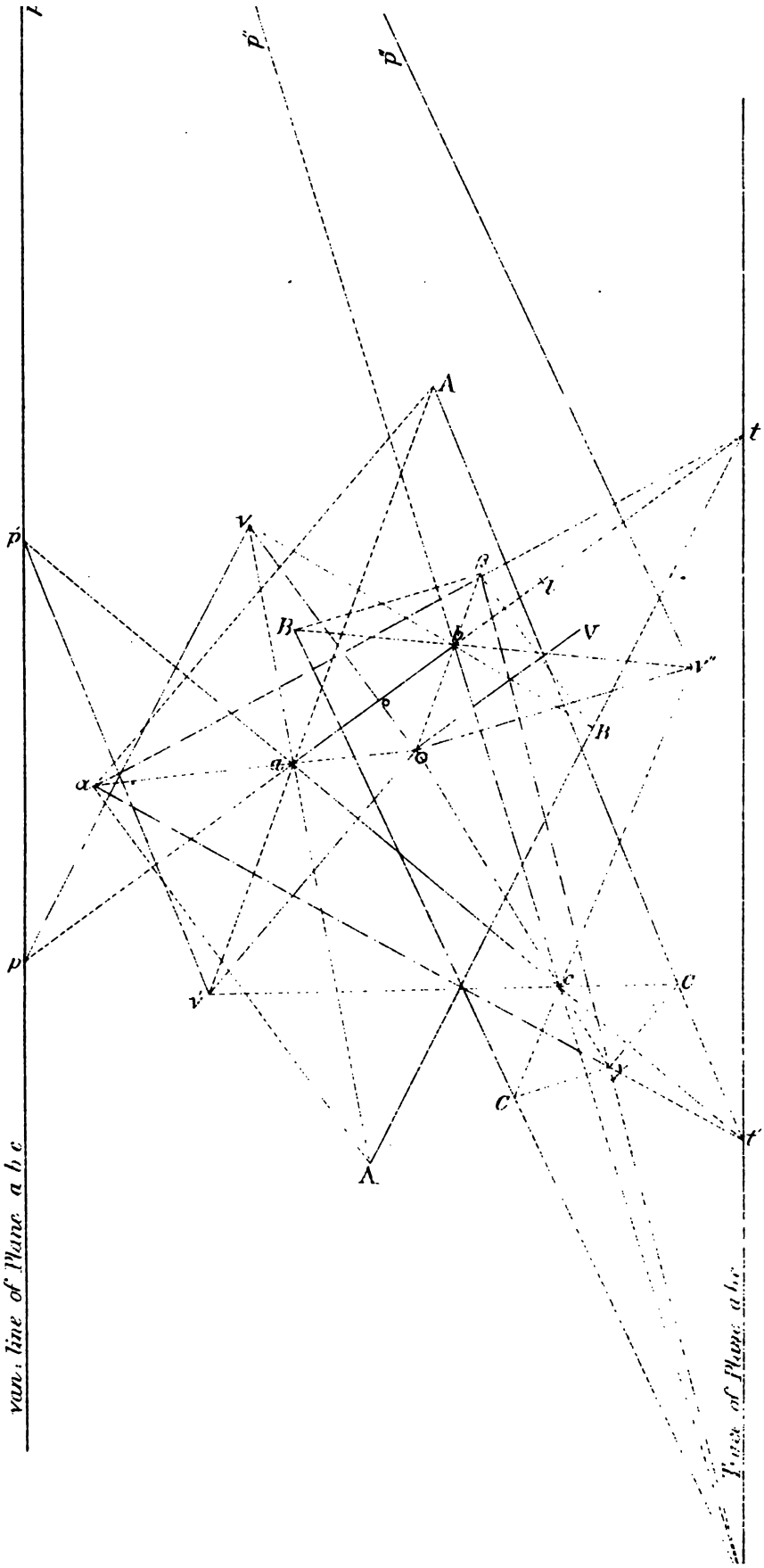
p'
p''
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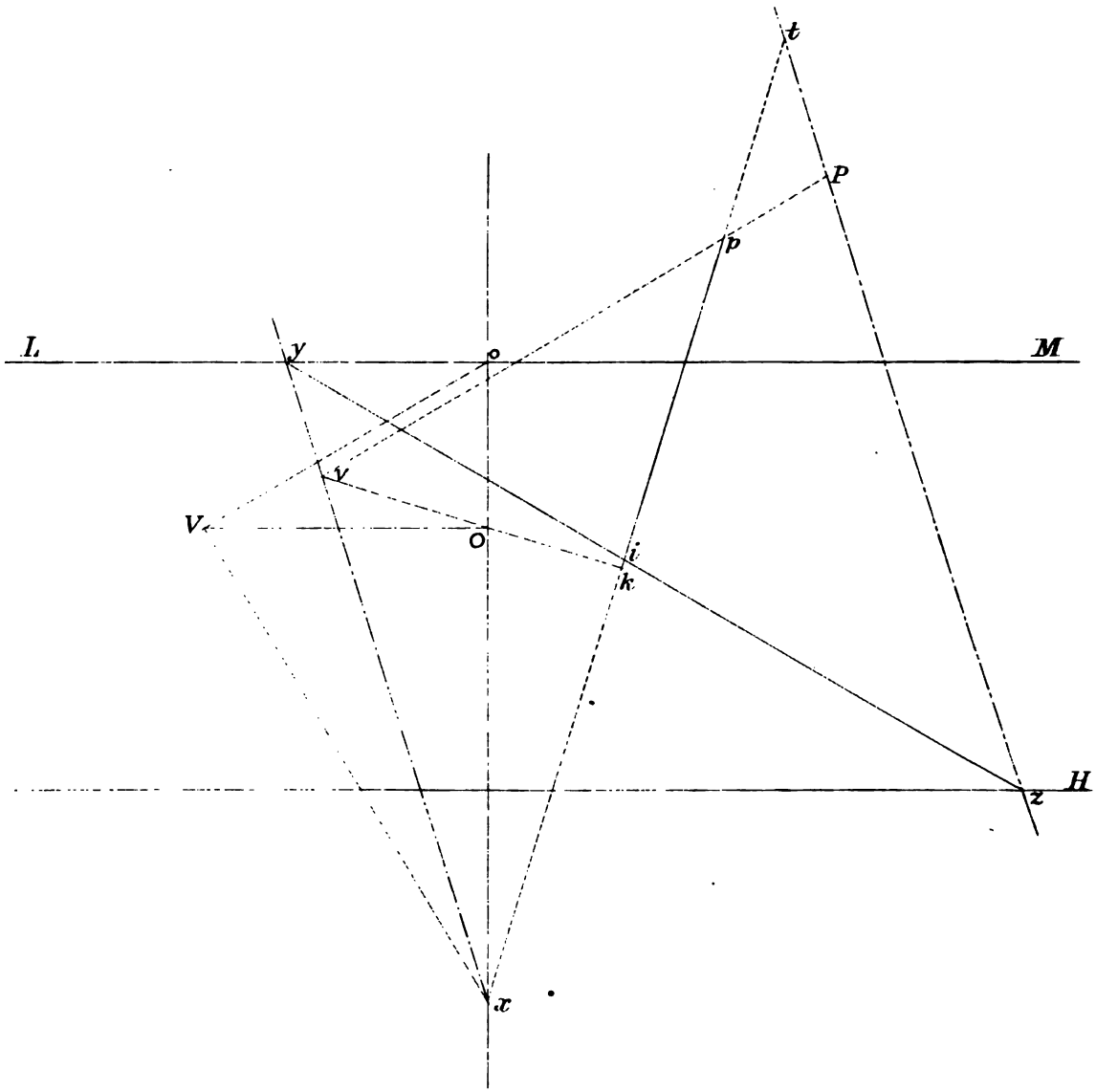
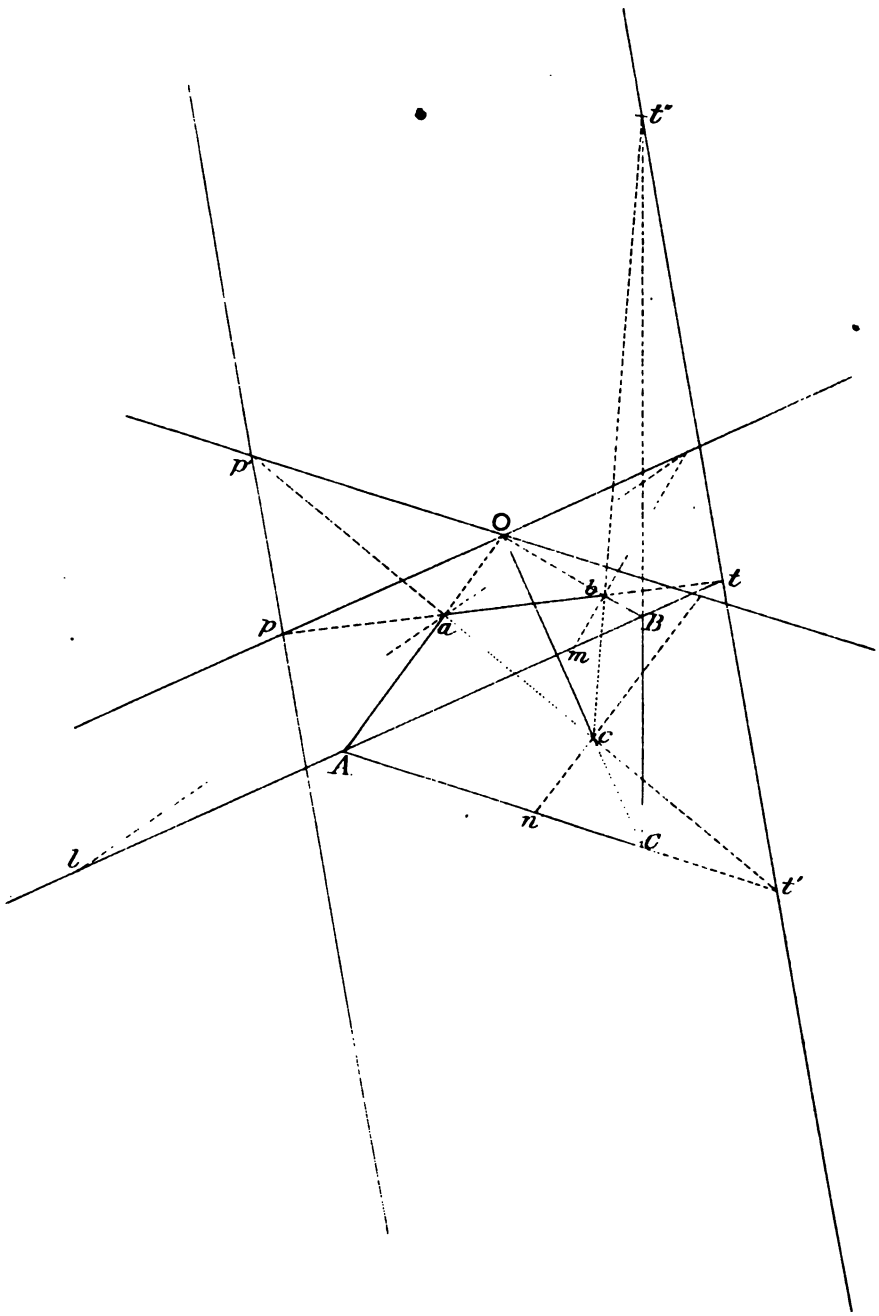
v'
Van. line of Plane a b c
Vert. of Plane a b c

Pl. XXIII.




t''
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t'

The point p is said to be given, when we know its projection and the length of its projector.

Find x, the vanishing point of perpendiculars to the given plane, and join $p\,x$, then $p\,x$ is the projection of the required perpendicular. Make $k\,v$ equal to the radial of the planes $x\,p$; join $v\,p$ and produce it to P, so that v P shall be equal to the projector of p. Through P draw P t parallel to $v\,x$. Then t will be the trace of $x\,p$. Through x and t draw $x\,y$, $t\,z$, parallel to each other, and cutting L M and G Z in y and z; $y\,z$ will be the projection of the intersection of the two planes (L M, G H), ($x\,y, z\,t$), and since the latter contains the perpendicular $p\,x$, the point i in which $y\,z$ cuts $p\,x$, will be the projection of the point in which the perpendicular $p\,x$ cuts the given plane.

45. (Plate XXIV.) Having given on the plane of projection the orthographic projection A B C of a triangle, and also the lengths of the perpendicular projectors, to determine the perspective projection of the triangle and the plane containing it.

Since the orthographic projectors are perpendicular to the plane of projection, their perspective representations will be A O, B O, and C O. Through O draw O v parallel to A B and equal to the distance of the vertex; and make A l, B m, equal to the real lengths of the perpendicular projectors, and join $l\,v$, $m\,v$, cutting A O, B O, in a and b; then $a\,b$ will be the perspective projection of that side of the triangle which has A B for its orthographic projection; produce $a\,b$ both ways to meet v O produced in p and A B in t; then p will be the vanishing point of $a\,b$, and t its trace. In like manner, the sides $b\,c$, $c\,a$, and their vanishing points p'', p' and traces t'', t' determined; and the lines drawn through t, t', t''; p, p', p'' will be the trace and vanishing line of the plane of the triangle $a\,b\,c$ required.

46. (Plate XXV.) To draw the projection of an octahedron, having given one of its faces $a\,b\,c$ and its plane (G H, E F).

Complete the hexagon of which $a\,b\,c$ are three angles; this is readily done by determining the vanishing point p of perpendiculars to lines l, the vanishing point of $b\,c$, and is effected by drawing the radial $v\,p$ perpendicular to $v\,l$; similarly, find the vanishing point q to the lines n, or those parallel to $a\,b$. Join $a\,p$, $q\,c$, and they will intersect in i the projection of the centre of the triangle $a\,b\,c$. Through b draw $b\,p$, $b\,q$ cutting $q\,i$, $p\,i$ in ϕ and ε; also join $c\,p$, $a\,q$ cutting each other in δ, and join $c\,\varepsilon$ and $a\,\phi$, and the hexagon will be completed. Now $\varepsilon\,\delta\,\phi$ is evidently the perspective of the orthographic projection of the face

of the octahedron, parallel to $a\,b\,c$, on the plane (G H, E F). Through $\delta, \varepsilon, \phi$, draw perpendiculars to the plane (G H, E F), by joining these points with x, the vanishing point of perpendiculars to this plane. Make $\delta\,d$ equal to the projection of the distance between two parallel faces of the octahedron. Join $l\,d$ to meet $\phi\,f$ in f; join $f\,m$ to meet $e\,e$ in e, and join $d\,e$, which will tend to the vanishing point n. Lastly, join $a\,d$, $d\,c$, $c\,e$, $e\,b$, $b\,f$, and $f\,a$, and the octahedron will be completed.

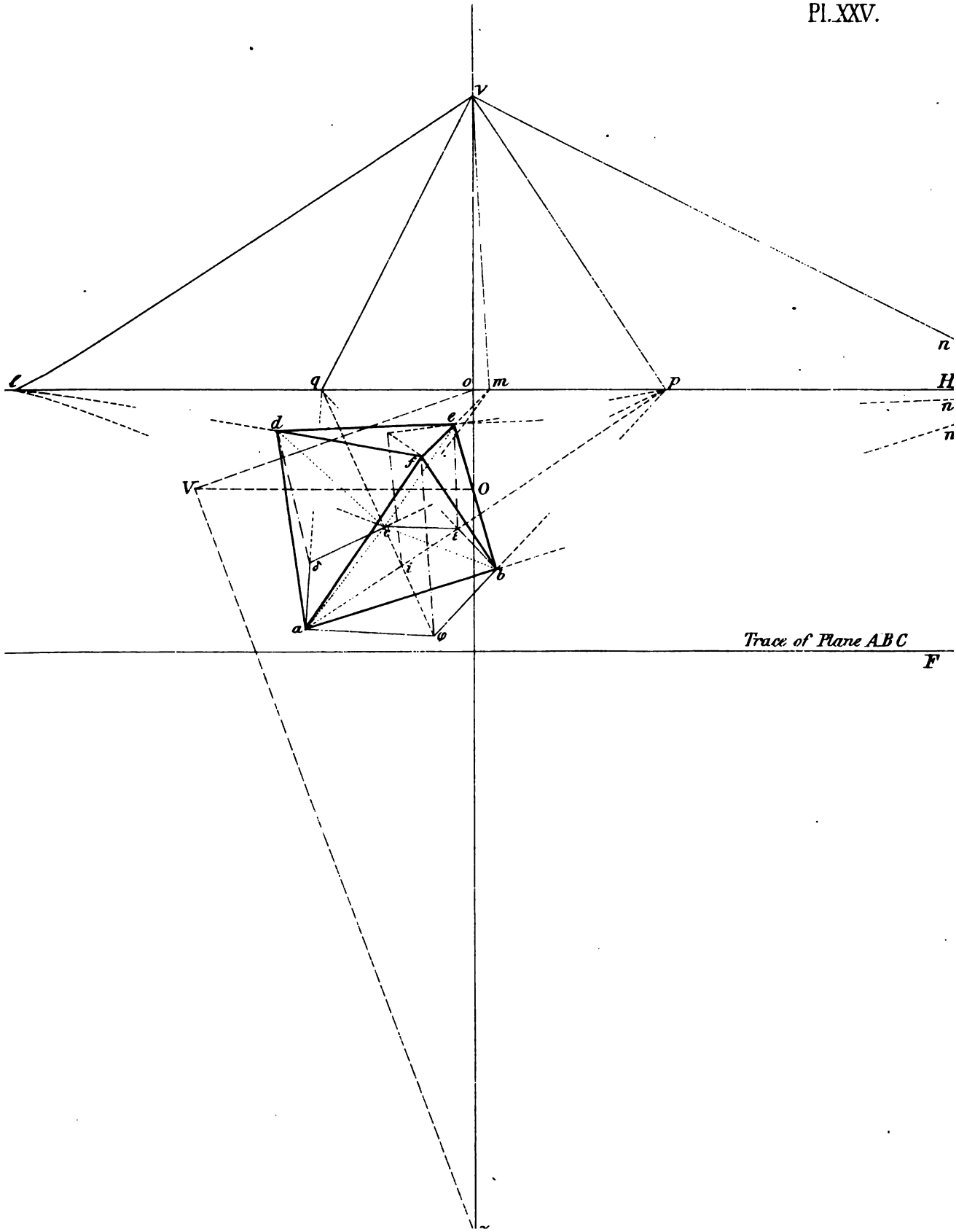
47. The projections of a dodecahedron and icosahedron, having given the projection and inclination of one face, are shown in Plates XXVI. and XXVII. The construction is precisely similar to that already described for projecting the octahedron, and needs no further explanation.

48. To draw the projection of a right cone, given the plane of its base, the projection $h\,i$ of one of its diameters, and the height of its axis. Also to find the shadow of the cone, on the plane of its base, from a given illuminating point r.

(Plate XXVIII.) Having determined the original H I of the given diameter $h\,i$, by turning the plane about its trace, describe the original circle on H I as diameter, and complete the projection of this circle by the method fully described in Article 28, (Plate V.) Through s, the projection of the centre of the circle, raise a perpendicular, $x\,s\,n$, to the plane of the base, and make $s\,n$ equal to the projection of the given height of axis; and from n draw tangents to the ellipse, and the projection of the cone will be complete. Lastly, to find its shadow from the given point r, from r drop a perpendicular on to the plane of the base, cutting it in w (Article 44, Plate XXIII.); join $r\,n$, $w\,s$ and produce them to meet in u; then u is the shadow of the vertex of cone, and tangents from u to the ellipse will be the shadow required.

49. Having given the projection $a\,i$ of an edge of a cone (a being a point in the base, and i the vertex of the cone), (Plate XXIX.), and the plane in which it lies tangential to the cone (A B, C D), and the direction $s\,i$ of the axis, to complete the projection.

Produce $a\,i$ to its vanishing point y, and since the plane containing $a\,i$, $s\,i$ is perpendicular to the tangential plane (A B, C D), the vanishing line of this plane must be $x\,y$, x being the vanishing point of perpendiculars to the plane (A B, C D). Produce $s\,i$ to its vanishing point u. Since the plane of the base of the cone is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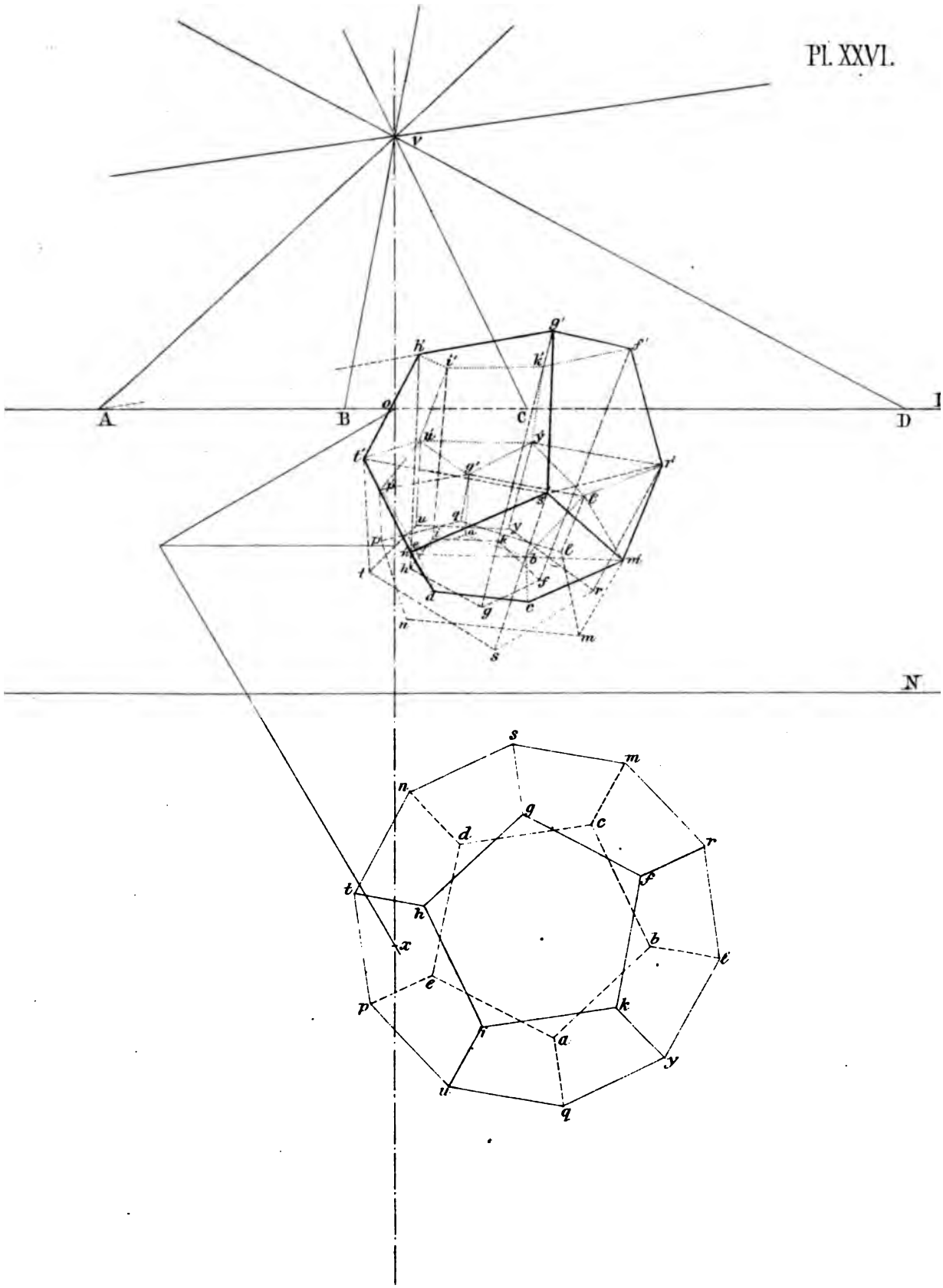
Trace of Plane ABC
F


V
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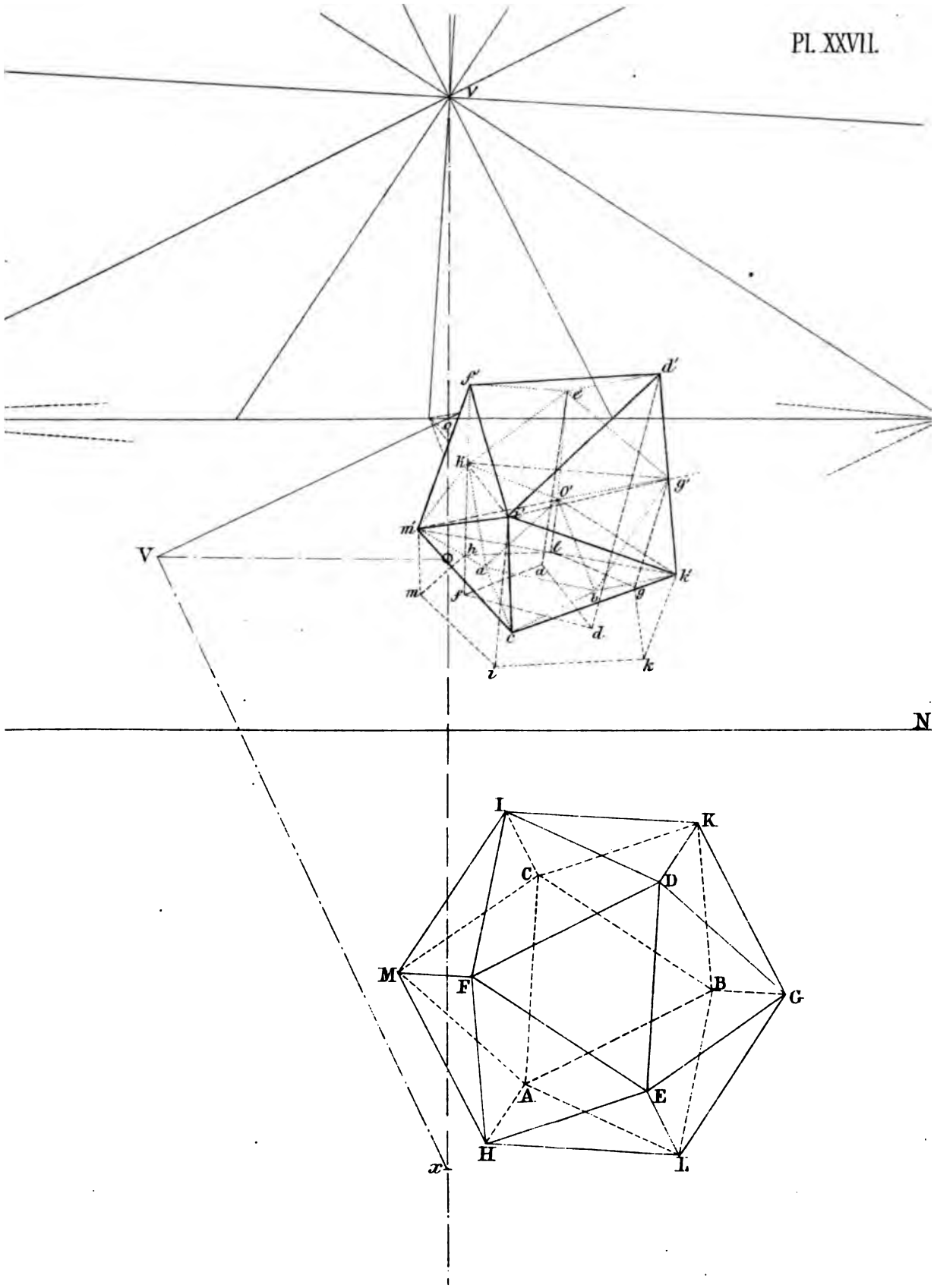
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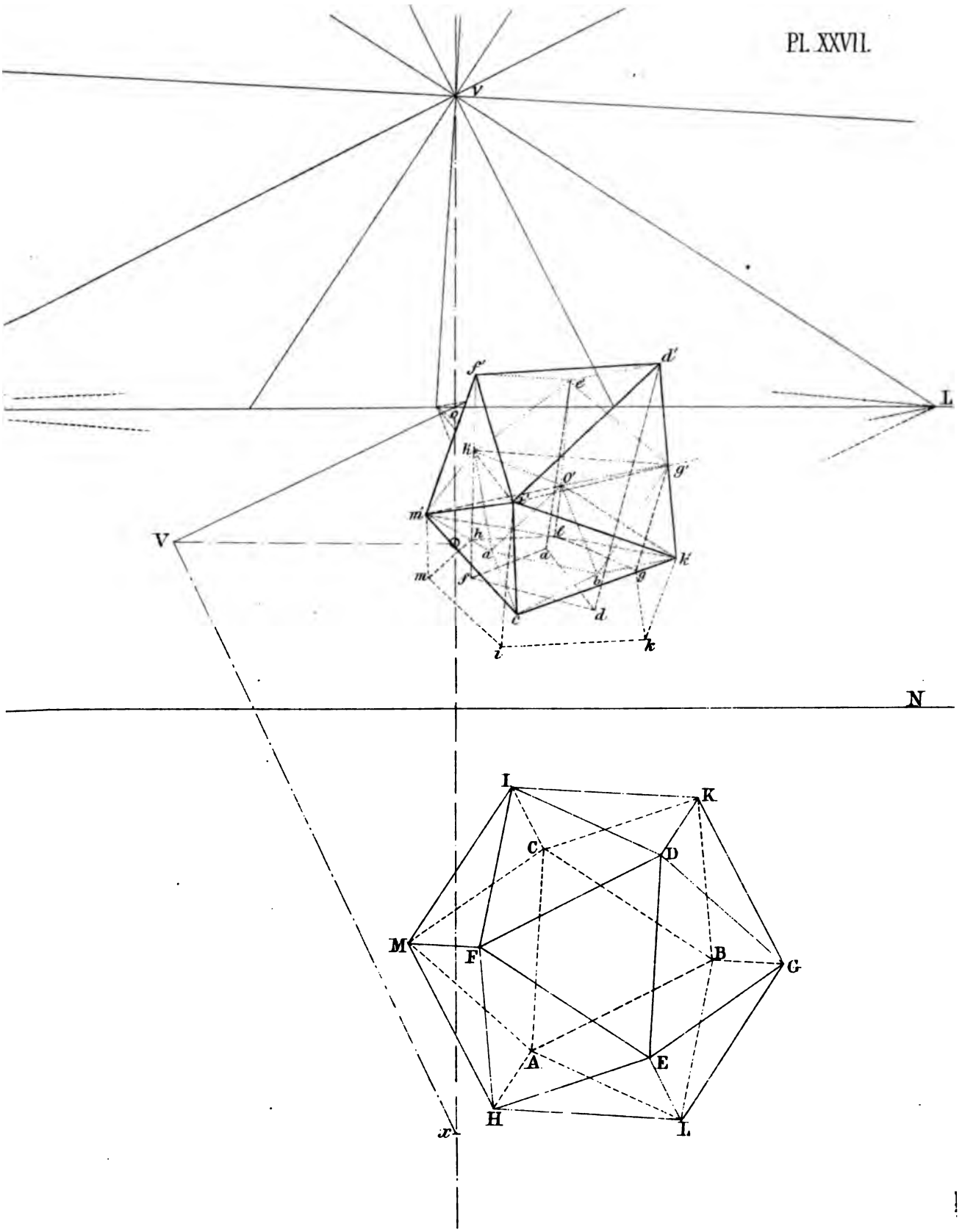
V
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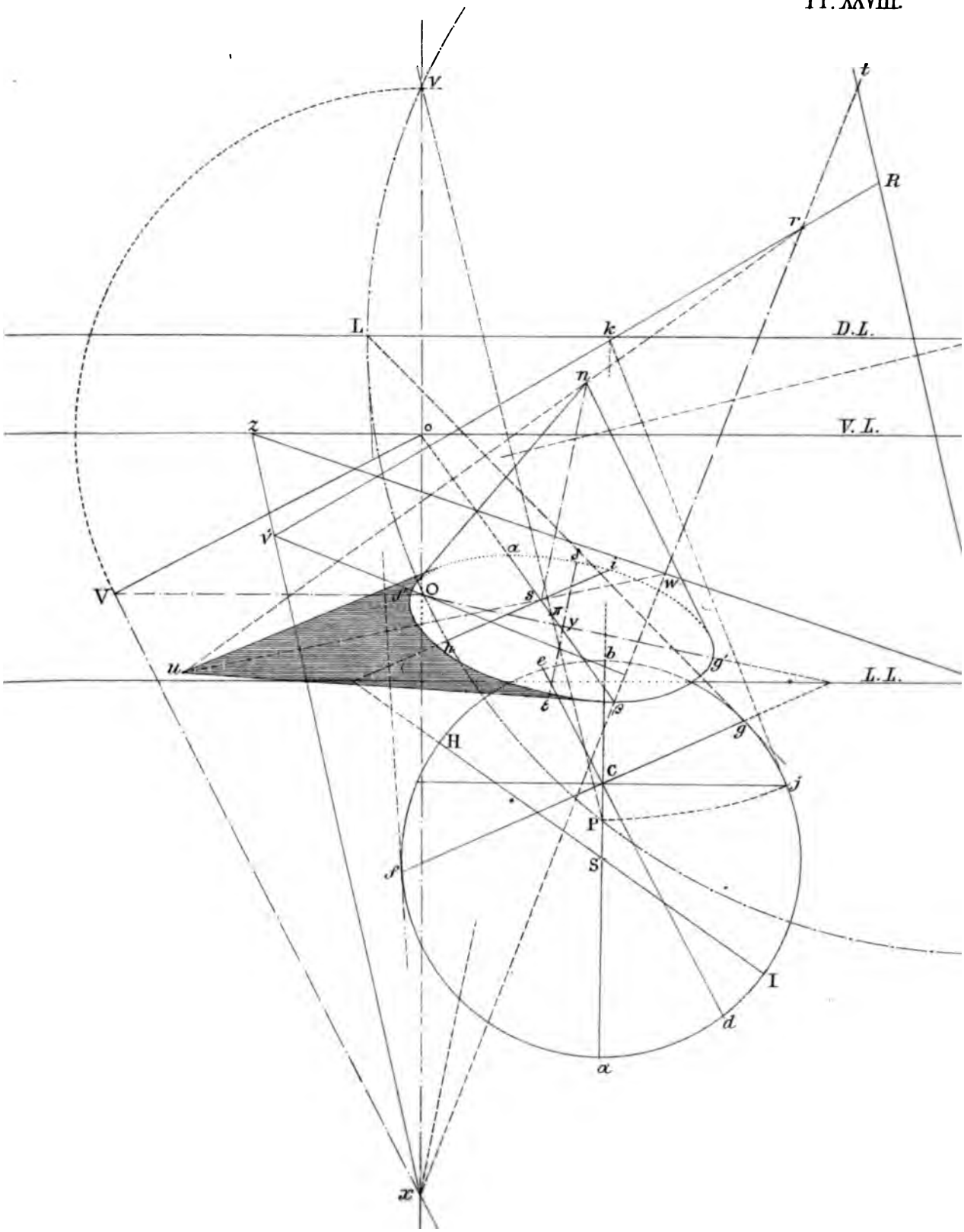
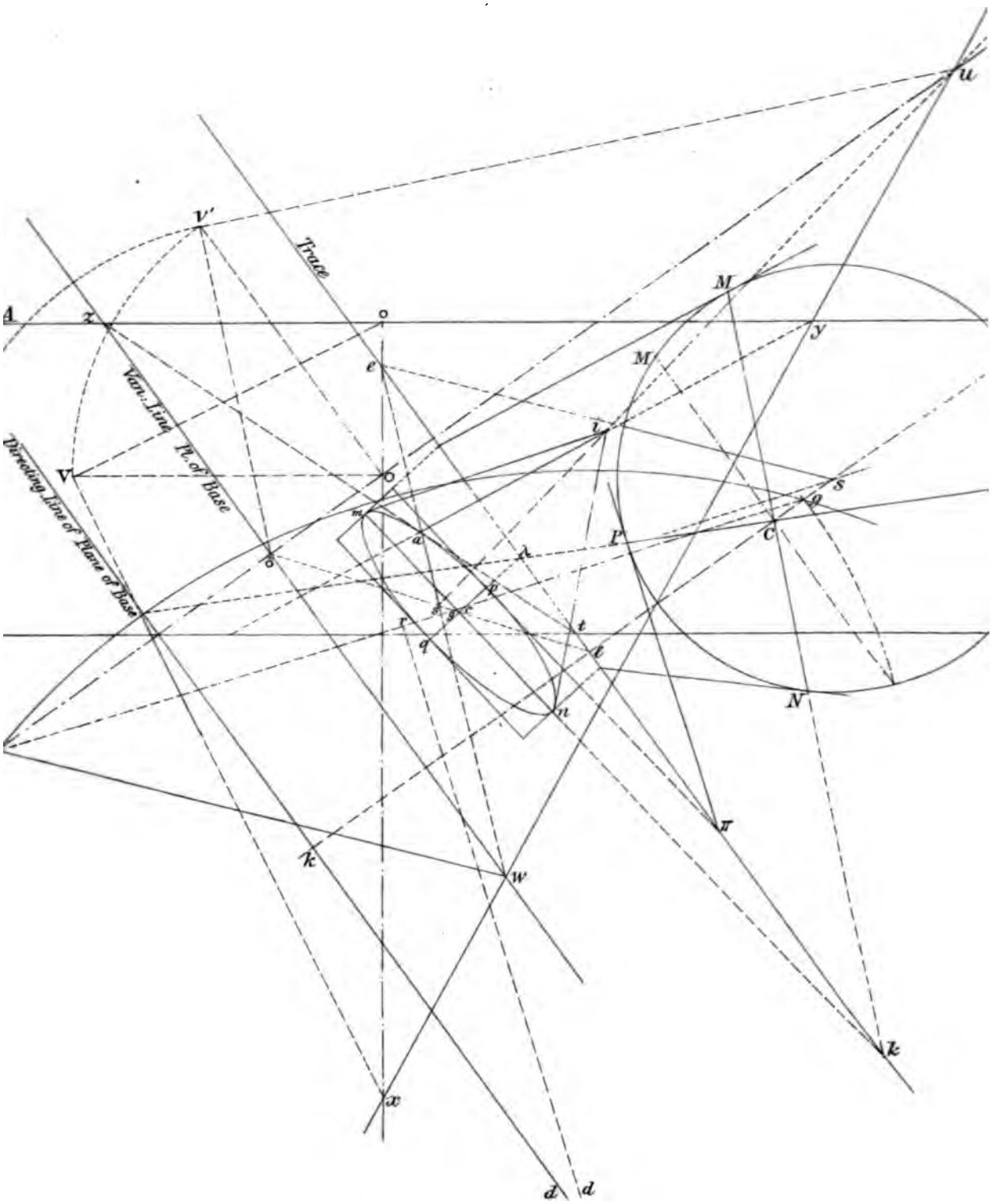
Pl. XXVIII.




Trace
Van. Line
Pt. of Base
Directing line of Plane of Base
V'
V
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u
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l'
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t
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k
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π
k
w
x
d
d

the vanishing line of a plane cutting the pyramid, and $y\,k$ its intersection with the plane of the base of pyramid. After having found t, the projection of p, on the plane of the base parallel to the cutting plane, draw $t\,g$ parallel to $y\,k$. Produce $b\,a$ to cut $y\,k, t\,g$, in h and g; join $p\,g$, and through h draw $h\,a\beta$ parallel to $p\,g$ cutting $p\,a, p\,b$ in a and β. Then $a\beta$ will be the intersection of cutting plane with the face $p\,a\,b$; produce $a\,f$ to meet $y\,k$ in i; join $i\,a$ cutting $p\,f$ in ϕ, and the intersection $a\,\phi$ will be determined; similarly the intersections with the remaining faces of the pyramid may be determined.

52. Having given two points a, b (Plate XXXII.), and a given plane D E F G, to find the vanishing and intersecting points of the line $a\,b$, and the point in which $a\,b$ meets the given plane.

Through O draw O k perpendicular to $a\,b$ and make $k\,v$ equal to the distance of the vertex from $a\,b$. Join $v\,b, v\,a$, and produce them to B and A, making v B, v A, equal to the given projectors of the points A, B. Join B A and produce it to meet $b\,a$ in t, and through v draw $v\,y$ parallel to B A to meet $a\,b$ in y. Then y will be the vanishing point of $a\,b$, and t its intersecting point or trace. To find the point in which $a\,b$ meets the given plane. Through y and t draw any plane $y\,z, t\,q$, cutting the given plane in the line $z\,q$, and the point p, in which $y\,t$ meets $z\,q$, will be the point required.

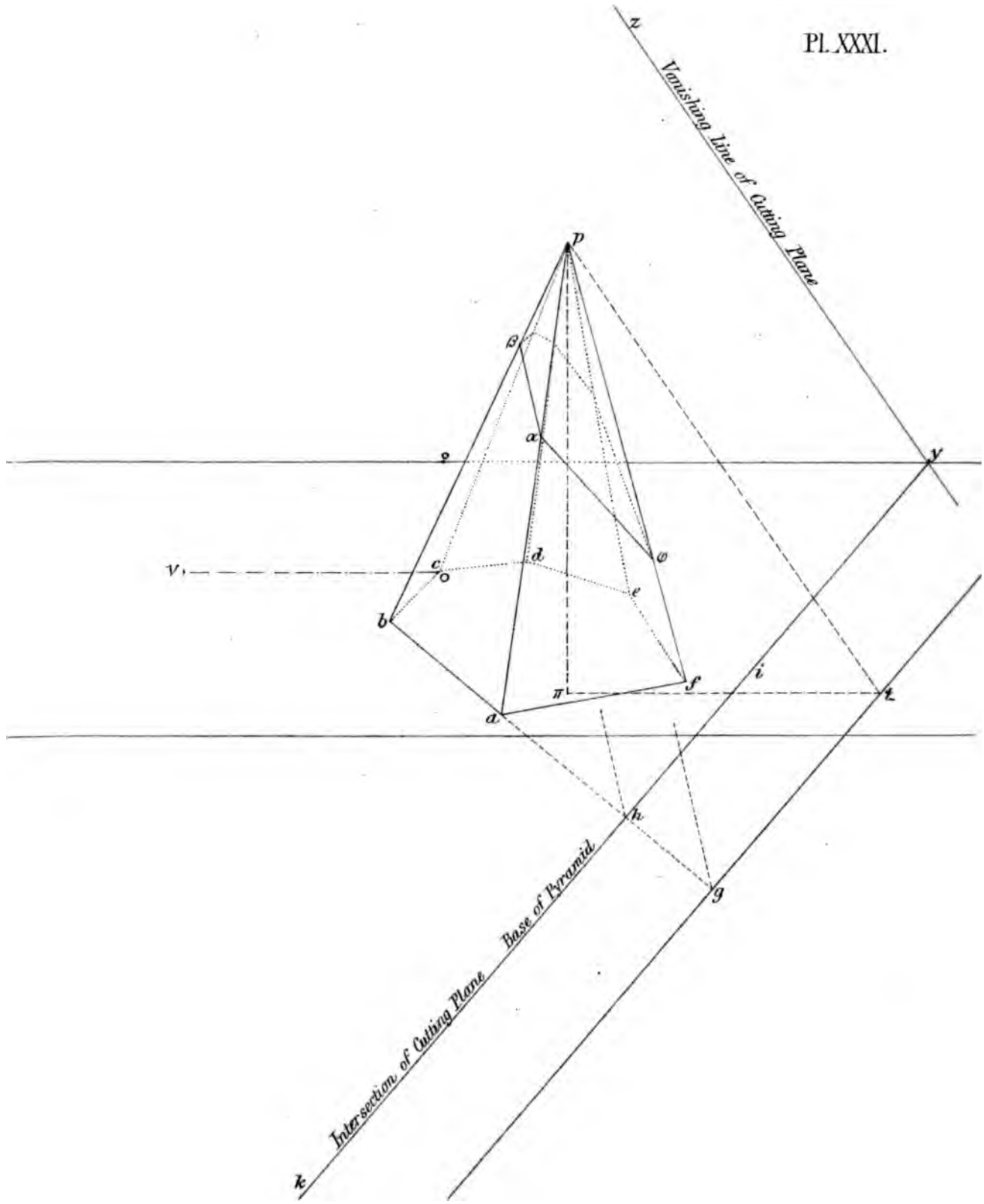
53. Having given an illuminating point s (Plate XXXII.) to find the shadow of a given line $y\,t$ lying in a given plane $y\,z, t\,q$, on another given plane D E F G. Find i, the parallel seat of s on the plane D E F G with respect to $y\,z, t\,q$. Join $i\,z, s\,y$, and produce them to meet in π. This point may be considered as the shadow of the point y, and, therefore, if $\pi\,p$ be joined the whole of the shadow is obtained.

The plane through s projecting the shadow is determined by the lines joining $m\,y, l\,t$.

It is obvious that the shadows of all lines in the plane $y\,z, t\,q$, which have y for their vanishing point, on the plane D E F G, must pass through π, the points s, i, y, and z, which determine this point remaining unchanged. Hence π may be called the focus of the shadows of the lines y, and $i\,z$ is therefore the locus of the foci of the shadows of all lines in the plane $y\,z\,t\,q$, on the plane D E F G.

If the illuminating point be at an indefinite distance from the vertex, as may be considered the case with respect to shadows cast by the sun, the projec-

PL. XXXI.


z
Vanishing Line of Cutting Plane
p
o
a
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y
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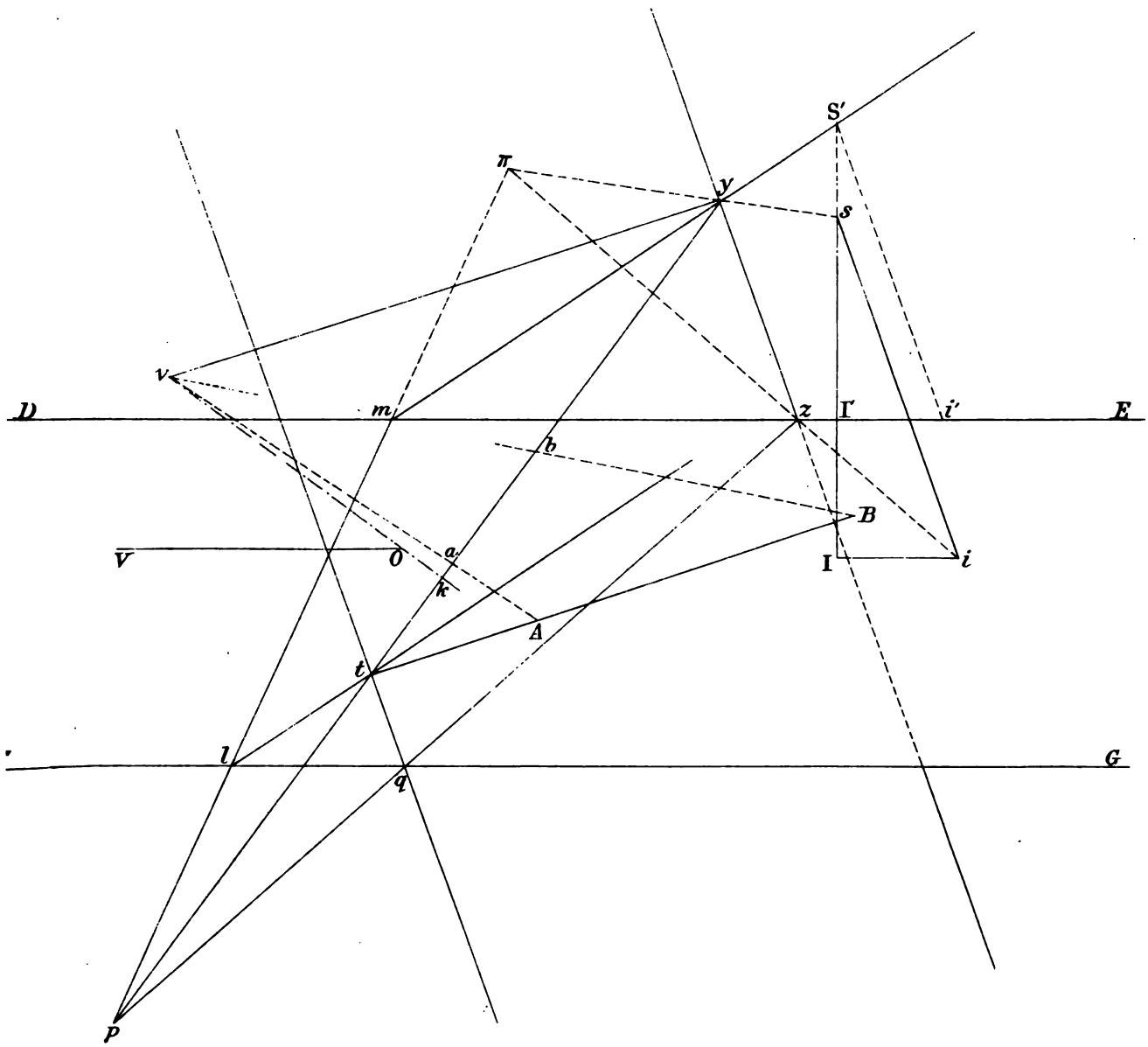
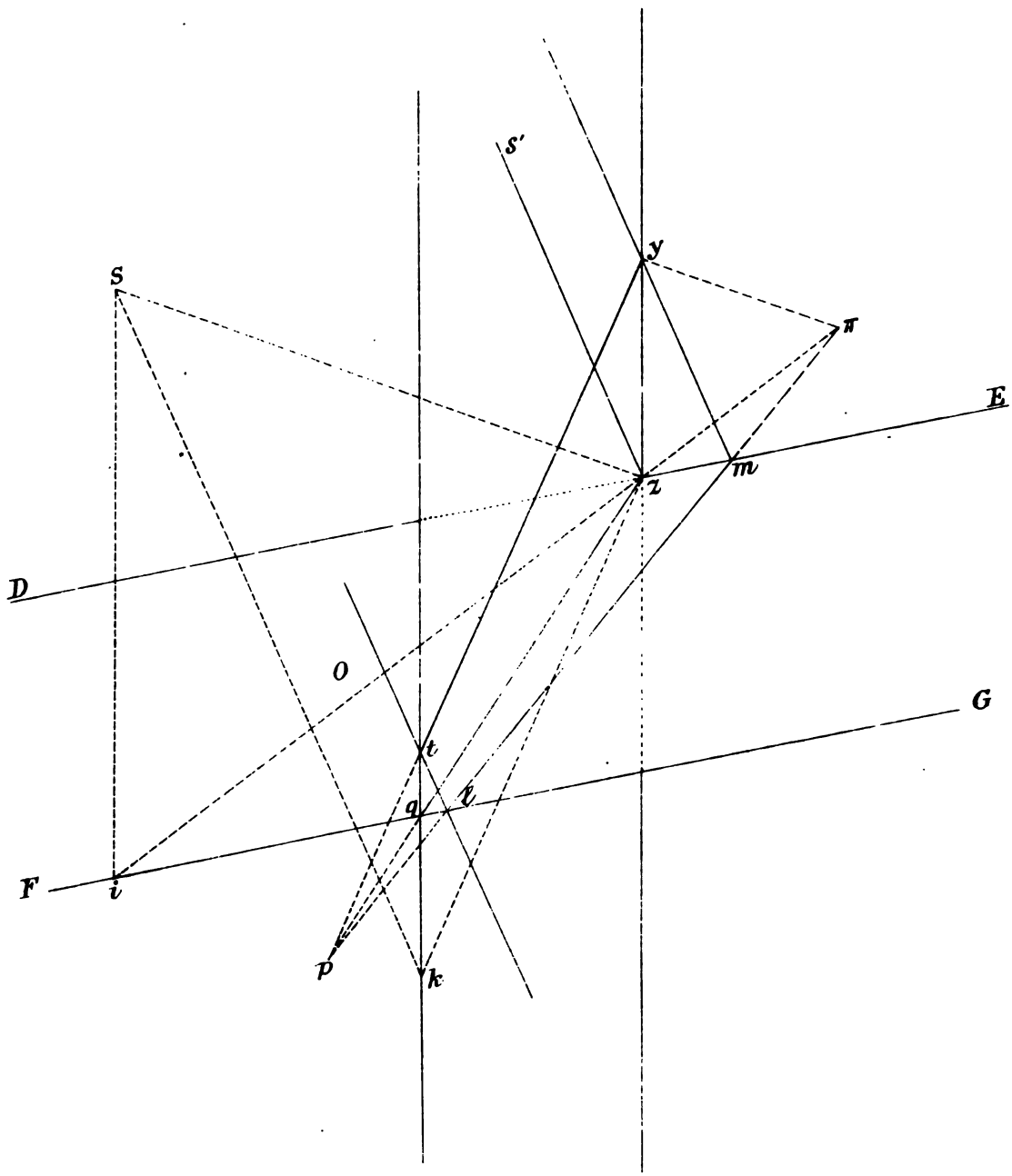
Intersection of Cutting Plane
Base of Pyramid

Pl. XXXII.



Pl. XXXIII.



tions of the imaginary seats of S′ (Plate XXXII.) will lie on the vanishing line D E; in this case the line joining S′ y will be the vanishing line of the plane projecting the shadow, and a line through t parallel to S′y, the intersecting line; whence the projection of the shadow $l\,m$ may be obtained. The focus π here coincides with the vanishing point m.

(Plate XXXIII.). We will next suppose the illuminating point s to lie in the directing plane, so that it can have no projection. To avoid confusion of lines, suppose the vertex to shift to z, everything remaining parallel; then the given vanishing lines $y\,z$, D E, will represent the parallels of the vertex of the respective planes, and $t\,q$, F G, their respective directing lines. Having given the position S of the illuminating point with respect to the shifted vertex z, draw $z\,k$ parallel to the given line $y\,t$, then $k\,z$ will be the director of the line $y\,t$, and k its directing point; therefore S k will be the directing line of the projecting plane, and, therefore, lines drawn through y and t, parallel to S k, will give the vanishing and directing lines of the projecting plane, whence the shadow $l\,m$ of the given line is determined. If S i be drawn parallel to $y\,z$, S z will be the director of the projecting rays, and $i\,z$ that of their seats on the given plane D E F G. Hence, if $y\,\pi$ be drawn parallel to S z to meet $i\,z$ in π, π will be the focus of all lines in the plane $y\,z$, $t\,p$ passing through y.

If the illuminating point (as S′) be at an infinite distance from the vertex in the directing plane, it is only necessary to draw through y and t lines parallel to the director S′ z of the projecting rays. In this case the focus π will coincide with the vanishing point m.

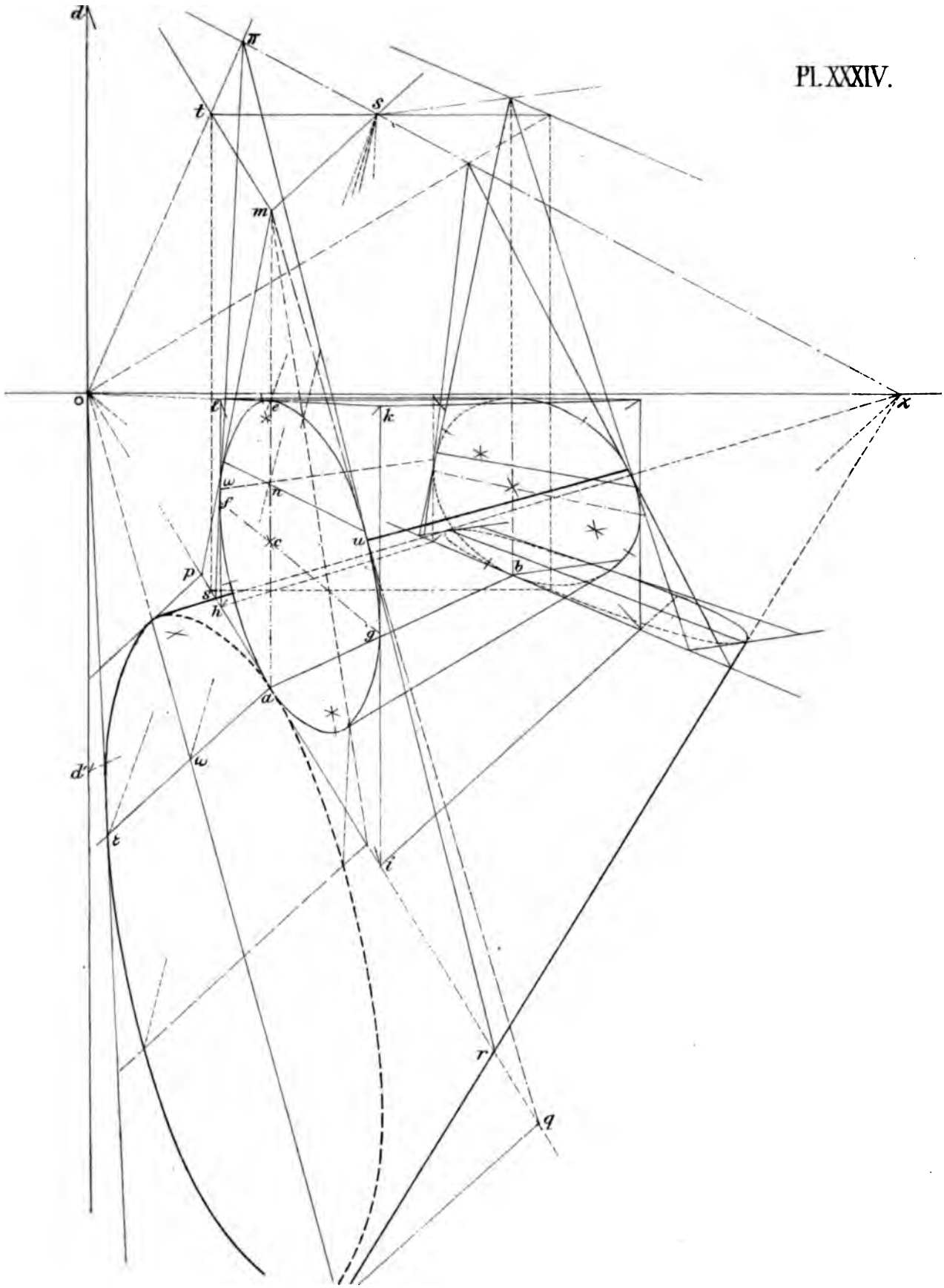
54. (Plate XXXIV.) A cylinder resting on a horizontal plane touches it along the given line $a\,b$, having its vanishing point in x; having given the radius of its circular end to complete the projection.

Find $d\,o\,d'$ the vanishing line of planes perpendicular to the lines x, and make $o\,d$, $o\,d'$ equal to the distance of the vertex. Draw $a\,c$ perpendicular to $o\,x$ and equal to the projection of the given vertical radius; and, by means of the vanishing point o and the points of distance d, d', complete the circumscribing square $h\,i\,k\,l$, touching the cylinder in the points f, g, a, and e, by means of which the ellipse may be drawn. Similarly, the projection of the other circular end of cylinder may be determined, and the common tangents to both ellipses will be the bonndaries of the solid, and tend towards x. Next, to determine the shadow of the solid from any illuminating point s, on the horizontal plane on which it rests. Find t, the projection of $s\,n$, the vertical plane $d\,o\,d'$, and parallel to the horizontal plane;

draw $t\,m$ parallel to $o\,a$, the intersection of the plane of end of cylinder and horizontal plane, and cutting $a\,c\,e$ produced in m; join $s\,m$. Then, because $t\,m$ is parallel to $o\,a$ and $s\,t$ to $o\,x$, $s\,m$ will be the projection of a line parallel to the horizontal plane, and therefore the point m of the line $a\,e$ can have no shadow on the horizontal plane. Hence the shadow of $a\,e$ will be a line through a parallel to $s\,m$. From m draw tangents to the ellipse cutting $o\,a$ in p and q, and lines through p and q will give the shadows of these tangents, and, having bisected $a\,\varepsilon$ in ω, join $o\,a$, $o\,\omega$, $o\,\varepsilon$, and a trapezium circumscribing the shadow is formed, whence the elliptical shadow may be drawn. To determine the boundaries of the light on the solid itself, having joined $o\,t$ and $x\,s$ produce them to meet in π, and from π draw tangents to the elliptical end touching it in u and ω, and meeting the line $o\,a$ in s and r. Then lines drawn from u, ω, to x will be the boundaries of light and shadow on the solid; and lines drawn through r and s to x will be the limit of the shadow on the horizontal plane, and will touch the elliptical shadows of the ends of the cylinder. The projection of the shadow of the further end of the cylinder is constructed in precisely the same way.

55. Having given the projection of a right cylinder and the plane of its base, to determine the projection of the section of the cylinder made by any given cutting plane.

(Plate XXXV.) Let $l\,u$ be the vanishing line of the plane of base $a\,c\,b\,d$ of cylinder, x the vanishing point of perpendiculars to the base or of the generating line of the cylinder. Let $y\,k$ be the vanishing line of the given cutting plane, and $y\,w$ its intersection with the plane of base $a\,b\,c\,d$. Draw $x\,u$ parallel to $y\,k$, meeting $l\,u$ in u, and draw the tangents $u\,c\,\mathrm{C}$, $u\,d\,\mathrm{D}$ to the base, and produce them to cut $y\,w$ in C and D; then, lines drawn through C and D parallel to $y\,k$ will cut the generating lines of the cylinder, $x\,c$, $x\,d$, produced, in γ and δ, two points of the required section, and $c\,\gamma$, $\mathrm{D}\,\delta$ will be tangents to the section at these points. Join $\delta\,\gamma$ to meet $y\,k$ in z. Also, join u and s, the apparent centre of the base, cutting the base in $b\,a$ and $y\,w$ in A; and having determined the points of section α and β by drawing $\mathrm{A}\,\alpha\,\beta$ parallel to $y\,k$, the lines joining $z\,\alpha$, $z\,\beta$ will be tangents to the section at these points, and the ellipse described in the trapezium formed by these four tangents will be the intersection sought. As many other points as may be necessary may be found in a similar manner by drawing chords to the base from u.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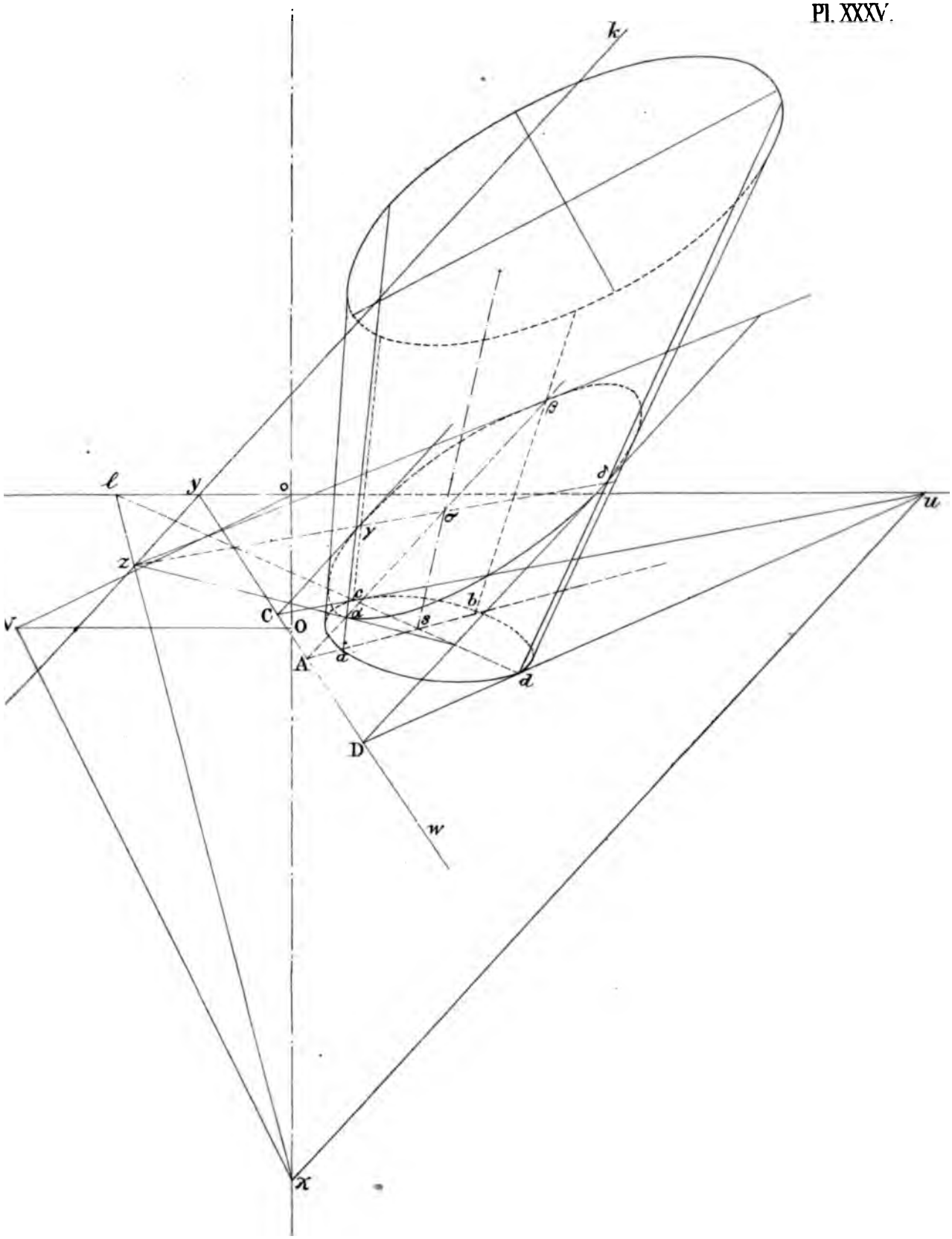
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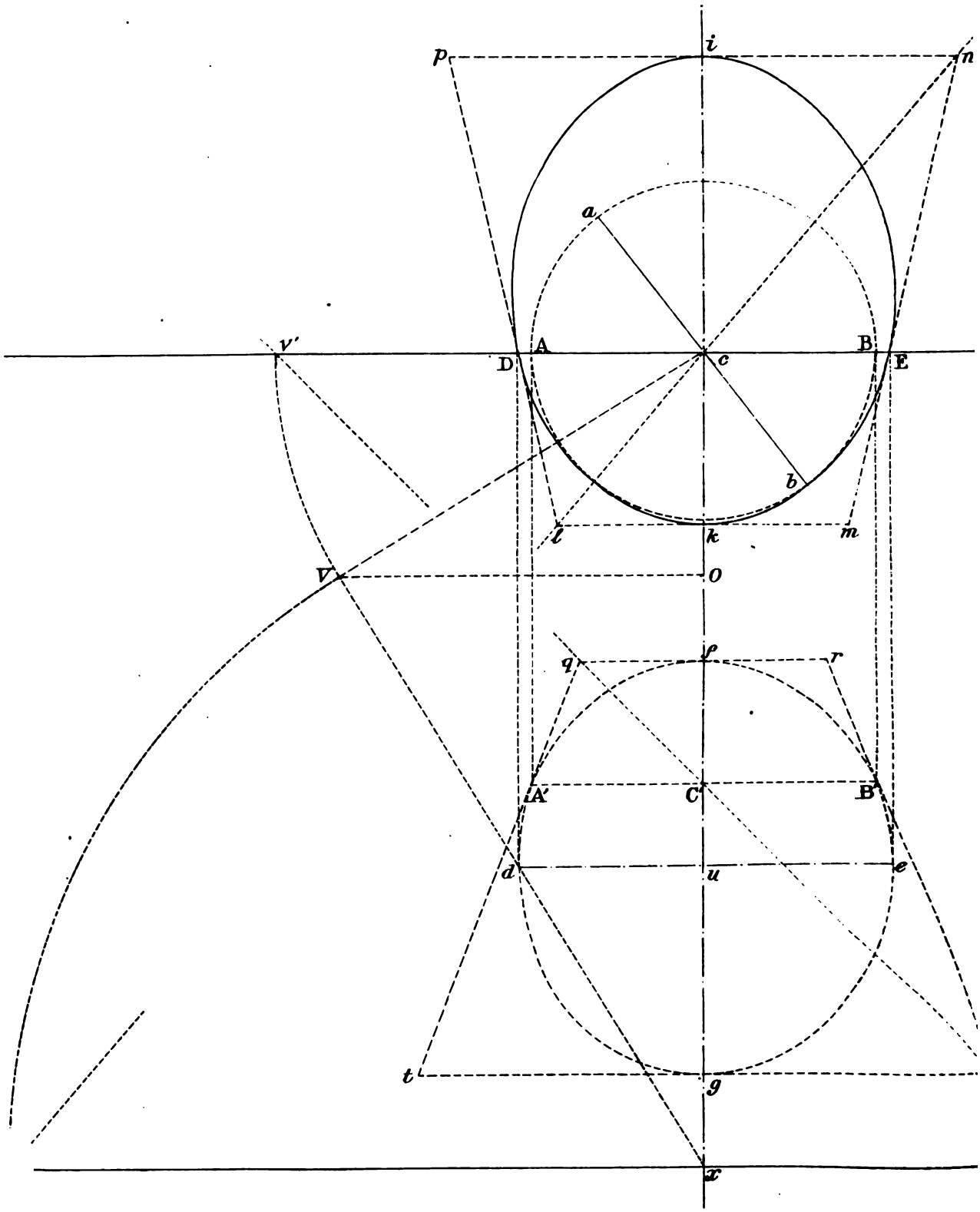
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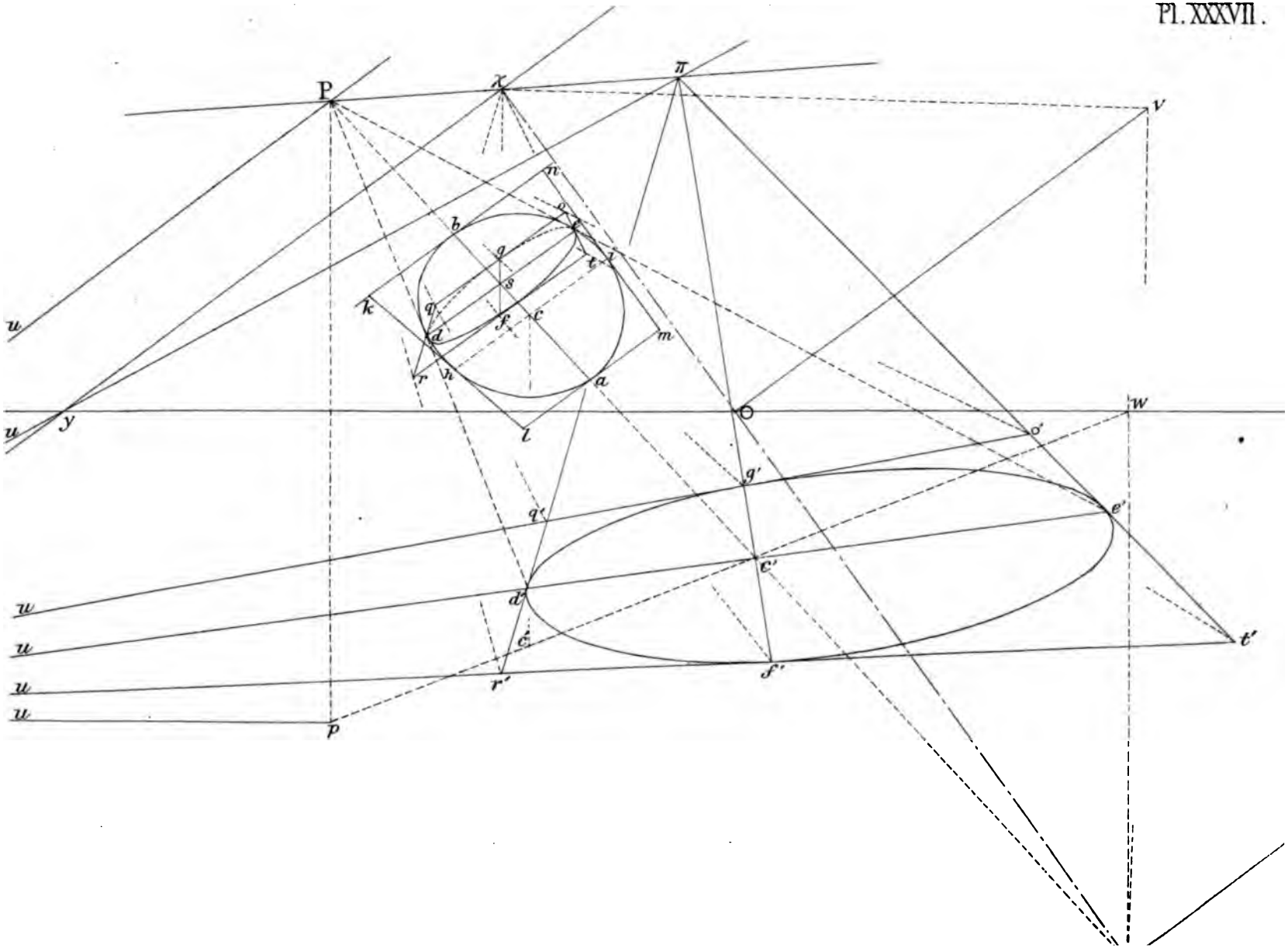
Pl. XXXVII.



56. To draw the projection of a sphere, having given the projection of the centre c (Plate XXXVI.) and the projection of the great circle of the sphere which is parallel to the plane of projection, and is therefore a circle, a A b B.

Join O c and draw A B through c perpendicular to O c, then A B is the vanishing line of planes parallel to that passing through the vertex and the centre of sphere, and therefore the projection of the great circle of the sphere, in which the plane through vertex and centre cuts the sphere, will lie wholly in the vanishing line A B, and c will be the projection of the centre of the circle of contact of the visual cone with sphere, as well as that of the centre of sphere. Suppose a cylinder to have the great circle projected in A B as base, and to have its axis c C parallel to the plane of projection, C being any point in the line c O produced. Draw A′ C B′ parallel to A B and equal to it, and on A′ B′ as diameter, describe the projection of a circle, lying in the parallel plane A B C. Bisect $f g$ in u, draw the axis $d e$, and draw d D, e E, parallel to c C, to meet A B in D and E; then D E will be the complete projection of the great circle of sphere perpendicular to that of the circle of contact, the projection of which is required and which constitutes the apparent outline of the sphere. Draw $x v''$ the vanishing line of planes perpendicular to A B and parallel to D E, and complete the projection of the circle on D E as diameter, and lying in the plane $c x$ V″. This ellipse will be the required outline of sphere.

57. To determine the shadow of a given sphere on a given plane from a given illuminating point. (Plate XXXVII.).

Let y O w be the given (horizontal) plane; P the given point, and p its vertical projection on the given plane; C the centre of the given sphere, and c its projection on the given plane.

Join P C and find its vanishing point z, which will lie in the line $w z$ at right angles to y O w, w being the vanishing point of $p c$. Draw $x y$, the vanishing lines of planes perpendicular to the lines z. Draw P e, P d tangents to the apparent contour of the sphere, and touching it in $d e$; join $d e$ and on $d e$ as a diameter describe the perspective of a circle lying in the plane $x y$, $d e$, by constructing the circumscribing square $q r$, $t o$. Then the projection of this circle $d g e f$ on the plane y O w, c will be the required shadow of the sphere. Find the point u, which is the projection of P on the given plane parallel to the plane $x y$. Join P x, $u y$, and produce them to meet in π. Then $y \pi$ is the locus of the foci of the shadows or projections of all lines in the plane $x y$, s on the plane y O w, c,

from the projecting or illuminating point P, and therefore π is the focus of the projections of the lines $r\,q, f\,g, t\,o$, which have x for their vanishing point. Hence, through c' the shadow of s, draw $c'\,\pi$, $c'\,u$, and complete the quadrilateral $o'\,q'\,r'\,t'$. This figure being treated as the perspective of a parallelogram lying in the plane $u\,\pi\,c'$, the perspective of the inscribed circle will be the shadow of the sphere required.

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